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Transitional Dynamics in a Tullock Contest with a General Cost Function*

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Abstract

This paper models an infinitely repeated Tullock contest in which two contestants contribute efforts to accumulate individual asset stocks over time. To investigate the transitional dynamics of the contest in the case of a general cost function, we linearize the model around the steady state. Our analysis shows that optimal asset stocks and their speed of convergence to the steady state crucially depend on the elasticity of marginal effort costs, the discount factor and the depreciation rate. We further analyze the effects of second prizes in the transition to the steady state as well as in the steady state itself. For a cost function with a constant elasticity of marginal costs, a lower discount factor, a higher depreciation rate and a lower elasticity imply a higher speed of convergence to the steady state. Moreover, a higher prize spread increases individual and aggregate asset stocks, but does not alter the balance of the contest in the long run. During the transition, a higher prize spread increases asset stocks and produces a more balanced contest in each period. Finally, a higher prize spread increases the speed of convergence to the steady state.

Keywords: Dynamic contest; transitional dynamics; logit contest; multiple prizes; rent-seeking

JEL Classification: C73, D72, L13

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1 Introduction

Competition is the essence of economics. Individuals and organizations compete for scarce goods, opportunities, positions, and status. Many of these competitions take the form of contests in which competitors make efforts by investing tangible and intangible resources and are rewarded based on their relative "efforts." In the context of business, for example, employees compete in promotion contests (Rosen, 1986 and Bognanno, 2001), firms compete in market share contests (Schmalensee, 1976; Piga, 1998) and R&D labs compete in patent race contests (Loury, 1970; Taylor, 1995). Competition in the form of contests, however, is not limited to the world of business. Contests can be observed in all fields of social life. Litigation (Wärneryd, 2000; Baye et al., 2005), rent-seeking (Farmer and Pecorino, 1999; Baye and Hoppe, 2003; Grossmann and Dietl, 2010), sport championships (Szymanski, 2003; Dietl et al. 2009), political campaigns (Glazer and Gradstein, 2005; Klumpp and Polborn, 2006), military conflicts (Garfinkel and Skaperdas, 2007), and many other forms of competitions take the form of contests.

Contests are usually modeled as static one-shot games. While this static approach may be sufficient to highlight many important aspects of contests, it ignores the fact that effort decisions in contests are often inter-temporarily connected. The effort invested in today's contest may affect the probability of winning tomorrow's contest. If a political party, for example, campaigns for electoral votes, it builds a political reputation that may affect not only this but also subsequent elections. Inter-temporal effects in contests have been analyzed in some specific economic fields. For example, Wirl (1994) presents a dynamic model in continuous time on lobbying. Moreover, Gradstein (1998) and Gradstein and Konrad (1999) study the design of multi-round contests with the elimination of the loser in each round, while Gurtler and Munster (2009) analyze sabotage in a tournament with two rounds. Leininger and Yang (1994) consider dynamic rent-seeking games with linear effort costs and show that wasteful expenditures in sequential-move games with an infinite time horizon are lower than in static simultaneous-move games.

Our paper analyzes the transitional dynamics in a classic Tullock contest with a general cost function. We further investigate the effect of second prizes on the transitional dynamics.[1] For this purpose, we consider a dynamic contest with an infinite time horizon, in which a Tullock contest is played every period by two contestants. In each period, both contestants simultaneously exert efforts by investing in some form of "asset," such as reputation, human capital, market share, prestige, weapons and so on. This asset stock depreciates over time and, in each period, determines the probability of winning an exogenous first prize (winner) and second prize (loser).[2]

[1]For a literature review about multiple-prize contests and the optimal allocation of prizes, see Sisak (2009).

[2]Note that a contest designer might be interested in the overall quality of the contest measured by the level of aggregate efforts. Simultaneously, a balanced contest could be another goal of the designer.

This paper contributes to the literature by showing how the effort cost function and second prizes affect the transitional dynamics. We extend the model of Yildirim (2005), who analyzes a two-stage contest with two players and a Tullock contest success function. In his model, the players have the possibility of adding to their previous efforts after having observed their rival's most recent efforts (Stackelberg game). We extend Yildirim's model to an infinitely-repeated contest and generalize the results for a general convex cost function. Contrary to Yildirim (2005), we allow the designer of the contest to offer first and second prizes instead of a single prize for the winner.

Our paper also complements the analysis by Szymanski and Valletti (2005), who model the incentive effects of contest prizes in a static game, along two dimensions. In contrast to Szymanski and Valletti (2005), we consider repeated instead of one-shot contests and we use a general cost function rather than linear costs. Moreover, we extend the paper by Grossmann et al. (2010), who analyze the investment behavior of clubs in a dynamic contest model of a professional team sports league. The authors focus on linear and quadratic costs as well as on the effect of revenue sharing on competitive balance. Their model shows that revenue sharing decreases competitive balance and the steady state is attained immediately if investment costs for playing talent are linear. Moreover, they find that revenue sharing decreases the speed of convergence to the steady state if investment costs for playing talent are quadratic. We generalize the model of Grossmann et al. (2010) by assuming a general convex cost function. In particular, we linearize the accumulation and Euler equation around the steady state. This approach allows us to investigate how the elasticity of marginal costs changes incentives to exert effort. To the best of our knowledge, the linearization procedure has not yet been applied to a Tullock contest model.

Our model shows that the speed of convergence crucially depends on the elasticity of marginal costs with respect to steady state efforts, the discount factor and the depreciation rate. We further investigate the effects of second prizes in the transition to the steady state as well as in the steady state itself.

The remainder of this paper is structured as follows. Section 2 introduces the model with its main assumptions, the optimality conditions and the steady state. In Section 3, we analyze the transitional dynamics of the model. Section 4 summarizes the main findings and concludes the paper.

For instance, in sports, a contest rather gains attention when the outcome is uncertain (Dietl, Lang, Rathke 2010).

2 Model

2.1 Notation and Assumptions

Consider an infinitely repeated Tullock contest in discrete time with two contestants. In each period $t \in \{0, ..., \infty\}$, each contestant $i \in \{1, 2\}$ expends irreversible efforts $e_{i,t} \in \mathbb{R}_0^+$ to accumulate an asset stock $E_{i,t} \in \mathbb{R}_0^+$.[3] Efforts are undertaken simultaneously, and the asset stock depreciates over time. The accumulation equation for the asset stock is given by[4]

$$E_{i,t} = (1 - \delta)E_{i,t-1} + e_{i,t}, \tag{1}$$

with $i \in \{1, 2\}$ and $t \in \{0, ..., \infty\}$. The parameter $\delta \in (0, 1)$ represents the depreciation factor. Equation (1) shows that efforts are necessary to maintain the existing asset stock. Before the competition starts, i.e., in period $t = -1$, each contestant i is assumed to have an initial asset stock given by $E_{i,-1} \in \mathbb{R}_0^+$.

In each period t, the available asset stock of contestant i determines which contestant wins the exogenously-given prize fund $V \in \mathbb{R}^+$, which is divided between the winner and the loser of the contest. We assume that the winner receives kV and the loser receives $(1 - k)V$ with $k \in (\frac{1}{2}, 1]$. That is, k is the fraction of the prize fund allocated to the first prize and $(2k - 1)$ characterizes the spread between first and second prize ("prize spread"). In the subsequent analysis, it holds that $i, j \in \{1, 2\},\ j \neq i$ and $t \in \{0, ..., \infty\}$, unless otherwise stated.

To calculate the probability $p_i \in [0, 1]$ that contestant i wins the first prize kV in period t, we utilize the Tullock contest success function (CSF) which is a widely-used functional form of a CSF in the contest literature. Its general form was provided by Tullock (1980) and axiomatized by Skaperdas (1996) and Clark and Riis (1998). The Tullock CSF is given by

$$p_i(E_{i,t}, E_{j,t}) = \begin{cases} \frac{E_{i,t}^\gamma}{E_{i,t}^\gamma + E_{j,t}^\gamma} & \text{if } \max\{E_{i,t}, E_{j,t}\} > 0 \\ \frac{1}{2} & \text{otherwise} \end{cases} \tag{2}$$

If contestant i does not win the first prize, it will win the second prize with certainty. Hence, the probability that contestant i wins the second prize $(1 - k)V$ is given by $(1 - p_i(E_{i,t}, E_{j,t}))$. The parameter $\gamma \in \mathbb{R}^+$ is called the "discriminatory power" of the CSF and reflects the degree to which the asset stock affects the winning probability.[5]

The ratio of the winning probabilities $p_i(E_{i,t}, E_{j,t})/p_j(E_{i,t}, E_{j,t})$ describes how even the contest is in period t. The contest is more balanced if the value of this ratio gets

[3] This asset stock can include reputation, human capital, market share, prestige, weapons and so on.
[4] Note that $E_{i,t}$ characterizes the state variable in our model.
[5] See, e.g., Dietl et al. (2008) for an analysis of the parameter γ in a static contest model. Moreover, Corchón and Dahm (2010) investigate foundations for prominent CSFs.

closer to one.

We assume that efforts of contestant i generate costs according to a (strictly) convex cost function $C(e_{i,t})$ with $C'(e_{i,t}) > 0$ and $C''(e_{i,t}) > 0$ for $e_{i,t} > 0$, $t \in \{0, ..., \infty\}$ and $C'(0) = 0$. Note that we concentrate on the effect of asymmetrical initial asset stocks on the optimal effort contributions of contestants over time and hence we consider contestants with a symmetrically, strictly convex cost function but asymmetrical initial asset stocks.

Contestant i's expected profits $\pi_{i,t}$ in period t is given by expected revenues minus costs:

$$\pi_{i,t}(e_{i,t}, E_{i,t}, E_{j,t}) = p_i(E_{i,t}, E_{j,t})kV + (1 - p_i(E_{i,t}, E_{j,t}))(1 - k)V - C(e_{i,t})$$

With probability $p_i(E_{it}, E_{jt})$, contestant i receives kV, and with probability $(1-p_i(E_{it}, E_{jt}))$ she/he receives $(1 - k)V$ at costs $C(e_{i,t})$. Future profits are discounted by a factor $\beta \in (0, 1)$. We further assume that the outside option for each contestant is zero.

2.2 Optimality Conditions and the Steady State

To solve the model, we follow the approach in Grossmann et al. (2010) and utilize the open-loop equilibrium concept. Under the assumption that efforts or investments are - at least to some degree - not measurable or observable by the competitors, the open-loop approach is the appropriate solution concept. If contestants were able to observe the opponent's effort level after each period, then the closed-loop approach would be appropriate. However, we believe that in many contest situations, contestants are not able to precisely evaluate the effort level chosen by their competitors such that the open-loop approach should be applied.[6]

Contestant i maximizes its expected discounted profits $\sum_{t=0}^{\infty} \beta^t \pi_{i,t}$ with respect to the stream $\{e_{i,t}\}_{t=0}^{\infty}$ and subject to the accumulation equation for the asset stock given by equation (1). We obtain the following Euler equation for contestant $i \in \{1, 2\}$:[7]

$$\frac{(2k-1)V\gamma E_{i,t}^{\gamma-1}E_{j,t}^{\gamma}}{(E_{i,t}^{\gamma} + E_{j,t}^{\gamma})^2} = C'(e_{i,t}) - \beta(1-\delta)C'(e_{i,t+1}). \tag{3}$$

Henceforth, variables without a time subscript indicate steady states. We derive the following results for a general, strictly convex cost function:

[6] For instance, suppose that a political party invests in lobbying activity for a political campaign. Even if these investments (i.e., the monetary inputs) are common knowledge among all contestants, some uncertainty regarding the actual effectiveness of the political campaign (i.e., the output) remains for all other contestants. In such a scenario, the open-loop approach seems to be more adequate. See Grossmann and Dietl (2009), who analyze the difference between closed-loop and open-loop equilibria in a two-stage contest model of a sports league.

[7] See Appendix A.1 for the derivation of the Euler equation, the steady state and the comparative statics.

1. In the steady state, it holds that $E_i = E_j \equiv E$ (implicitly defined by $(2k-1)\gamma V/(4E) = [1-\beta(1-\delta)]\, C'(\delta E)$) and $e_i = e_j \equiv e$ (implicitly defined by $e = \delta E$). It follows that $p_i(E_i, E_j) = p_j(E_i, E_j) = 0.5$ independently of initial asset stocks and the weight k attached to the first prize. Therefore, efforts and asset stocks are identical for both contestants in the steady state: that is, there is not only a relative convergence but also absolute convergence of the asset stocks in the long run if contestants have identical, strictly convex cost functions. This result holds even if contestants start with different initial asset stocks $E_{i,-1}$ and $E_{j,-1}$.

2. One can show that the steady state values E and e are increasing in the prize spread due to higher marginal benefits of effort. Therefore, if a contest organizer wants to increase (individual and aggregate) efforts or (individual and aggregate) asset stocks in the long run, she/he should increase the weight k attached to the first prize. However, the prize spread does not affect the balance of the contest in the long run because $p_i(E_i, E_j) = p_j(E_i, E_j)$ holds independent of the weight k attached to the first prize.

3. We find that a higher discount factor β implies a higher asset stock E in the steady state. Because future expected profits get less discounted, incentives for contestants to exert effort are higher such that also e increases in β. On the other hand, a higher depreciation rate δ reduces the incentives for the contestants to accumulate asset stocks in the steady state such that E decreases in δ. To observe the intuition behind this result, note that a higher depreciation rate has two effects on the steady effort $e = \delta E(\delta)$: (i) The steady asset stock $E(\delta)$ decreases in δ yielding lower steady state efforts e to maintain the asset stock. (ii) A higher depreciation rate implies that contestants must exert more effort to maintain the steady state asset stock. The second effect dominates the first effect such that steady state efforts e are increasing in the depreciation rate δ.

3 Transitional Dynamics

3.1 General Results

In this section, we investigate the dynamics of the model and analyze the transition (i.e., the short run) to the steady state. Because it is not possible to solve the model explicitly in the case of a general cost function, we linearize the model around the steady state. In particular, we linearize the accumulation equation (1) and the Euler equation (3) around the steady state. This procedure permits us to approximately determine the optimal path of the asset stocks for both contestants close to the steady state. It has an advantage in that we do not have to specify the cost function, but we are still able to provide an explicit path of the asset stocks.

By linearizing the asset stock accumulation equation (1) around the steady state, we obtain:

$$\hat{E}_{i,t+1} = (1-\delta)\hat{E}_{i,t} + \delta \hat{e}_{i,t+1}. \tag{4}$$

Henceforth, a variable $\hat{X}_t$ is defined as follows: $\hat{X}_t = dX_t/X = (X_t - X)/X$, where X represents the steady state value of the variable X_t. Hence, $\hat{X}_t$ represents the percentage deviation of X_t from the steady state value X.

By linearizing the Euler equation (3) around the steady state and using the results from Section 2.2, we obtain:

$$\hat{E}_{i,t} = \frac{\beta(1-\delta)\sigma(e)}{1-\beta(1-\delta)}\hat{e}_{i,t+1} - \frac{\sigma(e)}{1-\beta(1-\delta)}\hat{e}_{i,t}, \tag{5}$$

where

$$\sigma(e) \equiv \frac{eC''(e)}{C'(e)} \tag{6}$$

reflects the elasticity of marginal costs with respect to steady state efforts e. Using equations (4) and (5), which hold for contestant i and analogously for contestant j, we obtain the following system:

$$\underbrace{\begin{bmatrix} \hat{E}_{i,t+1} \\ \hat{E}_{j,t+1} \\ \hat{e}_{i,t+1} \\ \hat{e}_{j,t+1} \end{bmatrix}}_{\equiv \Gamma_{t+1}} = \underbrace{\begin{bmatrix} \frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + 1 - \delta & 0 & \frac{\delta}{\beta(1-\delta)} & 0 \\ 0 & \frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + 1 - \delta & 0 & \frac{\delta}{\beta(1-\delta)} \\ \frac{1-\beta(1-\delta)}{\beta(1-\delta)\sigma(e)} & 0 & \frac{1}{\beta(1-\delta)} & 0 \\ 0 & \frac{1-\beta(1-\delta)}{\beta(1-\delta)\sigma(e)} & 0 & \frac{1}{\beta(1-\delta)} \end{bmatrix}}_{\equiv Q} \underbrace{\begin{bmatrix} \hat{E}_{i,t} \\ \hat{E}_{j,t} \\ \hat{e}_{i,t} \\ \hat{e}_{j,t} \end{bmatrix}}_{\equiv \Gamma_t}$$

$$\Longrightarrow \Gamma_{t+1} = Q\Gamma_t \tag{7}$$

with $i, j \in \{1, 2\}$ and $j \neq i$. We define the matrix of Eigenvalues of Q as

$$\mu \equiv \begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & \mu_4 \end{bmatrix}$$

such that we obtain the decomposition $Q = P\mu P^{-1}$.[8] Hence, we conclude that

$$\Gamma_{t+1} = Q\Gamma_t \Leftrightarrow \underbrace{P^{-1}\Gamma_{t+1}}_{\equiv\tilde{\Gamma}_{t+1}} = \mu \underbrace{P^{-1}\Gamma_t}_{\equiv\tilde{\Gamma}_t} \Leftrightarrow \tilde{\Gamma}_{t+1} = \mu\tilde{\Gamma}_t.$$

[8]Without loss of generality, we assume that $|\mu_1| \leq |\mu_2|$. See Hamilton (1994) and Greene (2000) for a detailed review of the decomposition procedure.

Hence, we have generated canonical variables:

$$\begin{aligned} \tilde{\Gamma}_{t+1} &= \mu\tilde{\Gamma}_t \\ \begin{bmatrix} \tilde{\Gamma}_{1,t+1} \\ \tilde{\Gamma}_{2,t+1} \\ \tilde{\Gamma}_{3,t+1} \\ \tilde{\Gamma}_{4,t+1} \end{bmatrix} &= \begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & \mu_4 \end{bmatrix} \begin{bmatrix} \tilde{\Gamma}_{1,t} \\ \tilde{\Gamma}_{2,t} \\ \tilde{\Gamma}_{3,t} \\ \tilde{\Gamma}_{4,t} \end{bmatrix} = \begin{bmatrix} \mu_1\tilde{\Gamma}_{1,t} \\ \mu_2\tilde{\Gamma}_{2,t} \\ \mu_3\tilde{\Gamma}_{3,t} \\ \mu_4\tilde{\Gamma}_{4,t} \end{bmatrix} = \begin{bmatrix} \mu_1^{t+1}\tilde{\Gamma}_{1,0} \\ \mu_2^{t+1}\tilde{\Gamma}_{2,0} \\ \mu_3^{t+1}\tilde{\Gamma}_{3,0} \\ \mu_4^{t+1}\tilde{\Gamma}_{4,0} \end{bmatrix}. \end{aligned}$$

If $|\mu_1| < 1, |\mu_2| < 1, |\mu_3| > 1$ and $|\mu_4| > 1$, then $\tilde{\Gamma}_{3,0}$ and $\tilde{\Gamma}_{4,0}$ must be zero to satisfy the transversality condition.[9] Then, we are able to solve for the original variables $\Gamma_{1,t}$ and $\Gamma_{2,t}$ as follows:

$$\begin{aligned} \Gamma_0 &= P\tilde{\Gamma}_0 \\ \begin{bmatrix} \hat{E}_{i,0} \\ \hat{E}_{j0} \\ \hat{e}_{i0} \\ \hat{e}_{j0} \end{bmatrix} &= P \begin{bmatrix} \tilde{\Gamma}_{1,0} \\ \tilde{\Gamma}_{2,0} \\ \tilde{\Gamma}_{3,0} \\ \tilde{\Gamma}_{4,0} \end{bmatrix} = P \begin{bmatrix} \tilde{\Gamma}_{1,0} \\ \tilde{\Gamma}_{2,0} \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Hence, we get a system of four equations with four unknowns $\hat{e}_{i0}, \hat{e}_{j0}, \tilde{\Gamma}_{1,0}$ and $\tilde{\Gamma}_{2,0}$. Note that $\hat{E}_{i,0}$ and $\hat{E}_{j,0}$ are determined by $E_{i,-1}, E_{j,-1}, \hat{e}_{i0}$ and $\hat{e}_{j0}$. In particular, this computation determines the unique optimal time path of asset stocks and efforts. We can establish the following lemma:

Lemma 1 *Even if contestants have different initial asset stocks, there is a (locally) unique solution of contestant efforts and asset stocks in the linearized model.*

Proof. See Appendix A.2. ■

The lemma shows that there exists a unique solution of contestant efforts and asset stocks in the linearized model: that is, there is a unique path of efforts that is optimal for each contestant. As a result, we are able to negate the possibilities of "multiple equilibria" or "no equilibrium".

The dynamics of the state and control variables can be expressed as contingent on the stable Eigenvalues. We derive the stable solution of the linearized version of the model and we show that the dynamics of the model depend on the Eigenvalues of Q. First of all, we recover system (7) and redefine the matrix Q in the following way:

$$\Gamma_{t+1} = Q\Gamma_t \text{ with}$$

[9] The transversality condition in our model is different to the associated condition in standard growth models. In standard growth models, there must be zero capital in the long run: see, e.g., King et al. (1988). In our model, contestants have the following restriction: ex ante, expected profits must be positive for both contestants. Otherwise, it is not optimal for contestants to participate in the contest.

$$Q = \begin{bmatrix} \frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma}+1-\delta & 0 & \frac{\delta}{\beta(1-\delta)} & 0 \\ 0 & \frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma}+1-\delta & 0 & \frac{\delta}{\beta(1-\delta)} \\ \frac{1-\beta(1-\delta)}{\beta(1-\delta)\sigma} & 0 & \frac{1}{\beta(1-\delta)} & 0 \\ 0 & \frac{1-\beta(1-\delta)}{\beta(1-\delta)\sigma} & 0 & \frac{1}{\beta(1-\delta)} \end{bmatrix} \equiv \begin{bmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ m & 0 & f & 0 \\ 0 & g & 0 & h \end{bmatrix}$$

Using this notation in the system given by equation (7), we obtain

$$\begin{aligned} \hat{E}_{i,t+1} &= a\hat{E}_{i,t} + b\hat{e}_{i,t} \\ \hat{E}_{j,t+1} &= c\hat{E}_{j,t} + d\hat{e}_{j,t} \\ \hat{e}_{i,t+1} &= m\hat{E}_{i,t} + f\hat{e}_{i,t} \\ \hat{e}_{j,t+1} &= g\hat{E}_{j,t} + h\hat{e}_{j,t} \end{aligned}$$

with $i, j \in \{1, 2\}$ and $j \neq i$. Combining equations $\hat{E}_{i,t+1} = a\hat{E}_{i,t} + b\hat{e}_{i,t}$ with $\hat{e}_{i,t+1} = m\hat{E}_{i,t} + f\hat{e}_{i,t}$, we obtain the following second-order difference equation[10]

$$\begin{aligned} \frac{\hat{E}_{i,t+1} - a\hat{E}_{i,t}}{b} &= \hat{e}_{i,t} \\ \frac{\hat{E}_{i,t+2} - a\hat{E}_{i,t+1}}{b} &= m\hat{E}_{i,t} + f\frac{\hat{E}_{i,t+1} - a\hat{E}_{i,t}}{b} \\ 0 &= \hat{E}_{i,t+2} - (a+f)\hat{E}_{i,t+1} + (af - bm)\hat{E}_{i,t} \\ 0 &= [L^{-1} - (a+f) + (af - bm)L]\hat{E}_{i,t+1} \end{aligned} \tag{8}$$

The approximative dynamics are then summarized in Proposition 1.

Proposition 1 *In the case of a general strictly convex cost function, the approximative dynamics of efforts and asset stocks near the steady state for contestant $i = \{1, 2\}$ are summarized by equations*

$$\begin{aligned} e_{i,t} &= e + (\mu_s + \delta - 1)(E_{i,-1} - E)\mu_s^t, \\ E_{i,t} &= E + (E_{i,-1} - E)\mu_s^{t+1}, \end{aligned}$$

for all $t \in \{0, ..., \infty\}$, where the stable Eigenvalue $\mu_s \in (0, 1)$ of the linearized system is given by

$$\mu_s = \frac{1}{2}\left(\frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)} - \sqrt{\left(\frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)}\right)^2 - \frac{4}{\beta}}\right).$$

Proof. See Appendix A.3. ■

According to Proposition 1, we obtain an explicit optimal path for efforts and asset stocks through the linearization method. Contestant i's asset stock $E_{i,t}$ converges

[10] Note that L denotes a lag operator.

to the steady state asset stock E more quickly the closer the stable Eigenvalue of the linearized system is to zero. Hence, the stable Eigenvalue crucially determines the speed of convergence in the linearized model.

We can establish the following corollary which summarizes the main findings.

Corollary 1 *For a general, strictly convex cost function, we derive the following results:*

(i) Contestant i's efforts $e_{i,t}$ monotonically increase (decrease) over time into the steady state e if $E_{i,-1} > E$ ($E_{i,-1} < E$).

(ii) Lower initial asset stocks $E_{i,-1}$ imply higher initial efforts.

(iii) The speed of convergence of asset stocks is lower, the higher is the stable Eigenvalue μ_s of the linearized system.

Regarding Part (i), one can show that $(\mu_s + \delta - 1)$ is always smaller than zero. This implies that contestant i's efforts $e_{i,t}$ monotonically decrease over time into the steady state e if the initial asset stock is smaller than the steady state asset stock, i.e., $E_{i,-1} < E$. Otherwise, if $E_{i,-1} > E$, efforts $e_{i,t}$ monotonically increase over time into steady state efforts e.

Part (ii) shows that initial asset stocks critically influence the path of efforts and the asset stocks. For instance, suppose that $E_{i,-1} < E$, then a lower value of $E_{i,-1}$ implies higher initial efforts $e_{i,0}$, ceteris paribus.

Regarding Part (iii), one can see from equation $E_{i,t} = E + (E_{i,-1} - E)\mu_s^{t+1}$ that a lower (higher) stable Eigenvalue μ_s of the linearized system implies a higher (lower) speed of convergence of the asset stocks. In particular, the asset stocks immediately converge to the steady state value if the stable Eigenvalue converges to zero.

For a general convex cost function, the effect of the prize spread on the asset stocks during the transition and the speed of convergence is ambiguous. Note that $E_{i,t} = E + (E_{i,-1} - E)\mu_s^{t+1}$ for each $t \in \{0, ..., \infty\}$ during the transition. The individual and aggregate asset stocks depend on the prize spread because the steady state asset stock E depends on the weight k attached to the first prize (see Section 2.2). Furthermore, the stable Eigenvalue μ_s itself depends on (among other things) the elasticity of marginal costs $\sigma(e)$. According to Section 2.2, e itself depends on k as well. Therefore, it is not unambiguous, how the individual as well as aggregate asset stocks depend on the weight k attached to the first prize.

Similarly, for a general convex cost function, it is ambiguous how the elasticity of marginal costs $\sigma(e)$, the discount factor β and the depreciation rate δ affect the stable Eigenvalue. It follows that the effect on the speed of convergence of asset stocks is also not clear. To obtain further insights regarding the effects of the different parameters, we analyze a specific class of strictly convex cost functions in the next section.

3.2 Constant Elasticity of Marginal Costs

In this section, we analyze a cost function with a constant elasticity of marginal costs. One can show that the elasticity of marginal costs with respect to steady state efforts e denoted by $\sigma(e)$ is constant and equals η if and only if the cost function is given by

$$C(e_{i,t}) = \frac{\phi}{1+\eta} e_{i,t}^{1+\eta},$$

where $\phi \in \mathbb{R}^+$ and $\eta \in \mathbb{R}^+$ are constant parameters. In this case, the stable Eigenvalue $\mu_s \in (0,1)$ only depends on the constant parameters β, δ, η and is independent of the weight k attached to the first prize because the elasticity σ does no longer depend on the steady state efforts e.

3.2.1 The Effects of Elasticity of Marginal Costs, Discount Factor and Depreciation Rate

First, we show how the elasticity of marginal costs, the discount factor and the depreciation rate affect the speed of convergence of asset stocks to the steady state. Recall that the stable Eigenvalue of the linearized system crucially influences the speed of convergence. In particular, the speed of convergence is lower (higher), the higher (lower) is the stable Eigenvalue.

Computing the partial derivative of the stable Eigenvalue with respect to the elasticity σ yields

$$\frac{\partial \mu_s}{\partial \sigma} = \frac{\delta(1-\beta(1-\delta)}{2\sigma^2\beta(1-\delta)} \left(-1 + \frac{\sigma(1+\beta(1-\delta)^2) + \delta(1-\beta+\beta\delta)}{\sigma\beta(1-\delta)\sqrt{\left(\frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)}\right)^2 - \frac{4}{\beta}}} \right) > 0.$$

Hence, we conclude that a higher (lower) elasticity of marginal costs with respect to steady state efforts implies a higher (lower) Eigenvalue and therefore a lower (higher) speed of convergence of the asset stocks. This result is because a high elasticity implies a more sharply curved cost function which makes it rather profitable to smooth efforts over time. If the elasticity is rather low, high efforts in the beginning of the contest are profitable such that there is lower smoothing behavior over time, and hence, convergence occurs much faster. In the limiting case of a linear cost function, the steady state asset stock E_i for contestant i would be achieved immediately in the first period independent of initial asset stocks.

How do the other parameters influence the stable Eigenvalue of the linearized system and therefore the speed of convergence? We find that a higher discount factor β and/or a lower depreciation rate δ imply a higher Eigenvalue, i.e., $\partial \mu_s / \partial \beta > 0$ and $\partial \mu_s / \partial \delta < 0$.[11] It follows that a higher (lower) β yields a lower (higher) speed of convergence of the asset stocks. Conversely, the speed of convergence of the asset stocks is higher (lower), the higher (lower) is the depreciation rate δ. If the discount factor is low, the future is less important because future expected profits get more discounted and therefore convergence occurs faster. In the limiting case of $\beta = 0$, the steady state would be immediately attained in the first period. On the other hand, if the depreciation rate is high, a lower percentage of the current asset stock is taken over to the next period and the speed of convergence is higher. In the limiting case of $\delta = 1$, the whole asset stock is depreciated and convergence to the steady state occurs again in the first period.

The following corollary summarizes our results:

Corollary 2 *For a cost function with a constant elasticity of marginal costs, the speed of convergence of asset stocks is lower:*
(i) the higher is the elasticity of marginal costs, (ii) the higher is the discount factor, and (iii) the lower is the depreciation rate.

3.2.2 The Effect of the Second Prize

In this section, we investigate the effect of the second prize on the asset stocks in the transition to the steady state and on the speed of convergence. First, we analyze how the prize spread affects the asset stocks in the transition. From Section 2.2, we know that $\partial E / \partial k > 0$. Therefore,

$$E_{i,t} = E + (E_{i,-1} - E)\mu_s^{t+1} = E_{i,-1}\mu_s^{t+1} + (1 - \mu_s^{t+1})E$$

is increasing in k because $\partial E_{i,t} / \partial k = (1 - \mu_s^{t+1})\partial E / \partial k > 0$, $i = \{1, 2\}$. Then, we also derive that the aggregate asset stock $E_{i,t} + E_{j,t} = (E_{i,-1} + E_{j,-1})\, \mu_s^{t+1} + 2(1 - \mu_s^{t+1})E$ is increasing in k because $\partial(E_{i,t} + E_{j,t})/\partial k = 2(1 - \mu_s^{t+1})\partial E / \partial k > 0$ with $i, j \in \{1, 2\}$ and $j \neq i$.

Moreover, we can show that the contest becomes more balanced if the weight k attached to the first prize increases:

$$\frac{\partial \left(\frac{p_{i,t}}{p_{j,t}} \right)}{\partial k} = \gamma \left(\frac{E + (E_{i,-1} - E)\mu_s^{t+1}}{E + (E_{j,-1} - E)\mu_s^{t+1}} \right)^{\gamma - 1} \frac{(1 - \mu_s^{t+1})\mu_s^{t+1}(E_{j,-1} - E_{i,-1}) \overbrace{\frac{\partial E}{\partial k}}^{>0}}{(E + (E_{j,-1} - E)\mu_s^{t+1})^2}$$

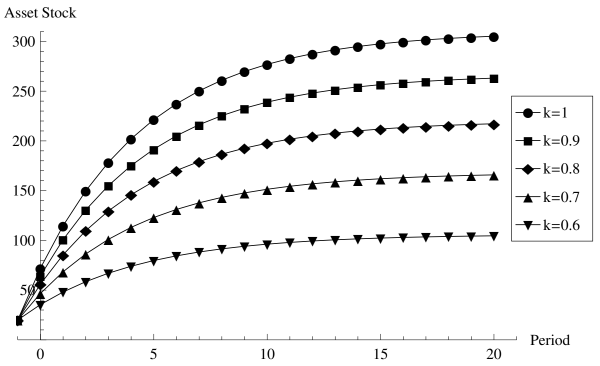
$$\Longrightarrow \quad \frac{\partial \left(\frac{p_{i,t}}{p_{j,t}} \right)}{\partial k} \begin{cases} > 0 \text{ if } E_{j,-1} > E_{i,-1} \\ = 0 \text{ if } E_{j,-1} = E_{i,-1} \\ < 0 \text{ if } E_{j,-1} < E_{i,-1} \end{cases}$$

[11] Similar to the elasticity, it is easy to show that the partial derivative of μ_s with respect to β is always positive, while the corresponding derivative with respect to δ is always negative.

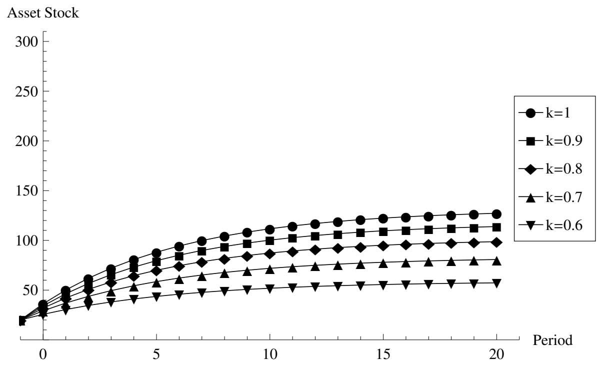
with $i, j \in \{1, 2\}$ and $j \neq i$. Suppose that $E_{j,-1} < E_{i,-1}$, then $p_{i,t}/p_{j,t} > 1$. Since $\partial\left(p_{i,t}/p_{j,t}\right)/\partial k < 0$, we know that the balance of the contest is increasing in k. Suppose that $E_{j,-1} > E_{i,-1}$, then $p_{i,t}/p_{j,t} < 1$. Since $\partial\left(p_{i,t}/p_{j,t}\right)/\partial k > 0$, we know that the balance of the contest is increasing in k. If $E_{j,-1} = E_{i,-1}$, then $p_{i,t}/p_{j,t} = 1$. Since $\partial\left(p_{i,t}/p_{j,t}\right)/\partial k = 0$, we know that the balance of the contest is not affected by changing k.

In the case of a cost function with a constant elasticity of marginal costs, we are able to explicitly determine the optimal asset stock path for each contestant through a simulation. Suppose that $\delta = 0.1, \beta = 0.95, \gamma = 1, V = 1000, \phi = 1, E_{i,-1} = 20$ and $E_{j,-1} = 5$. We vary η and k and obtain Figures 1a and 1b for contestant i and Figures 1c and 1d for contestant j.

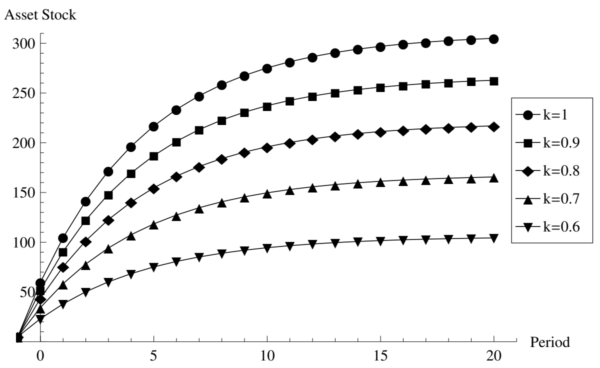
Figure 1: Second Prize Effect on Asset Stock Convergence



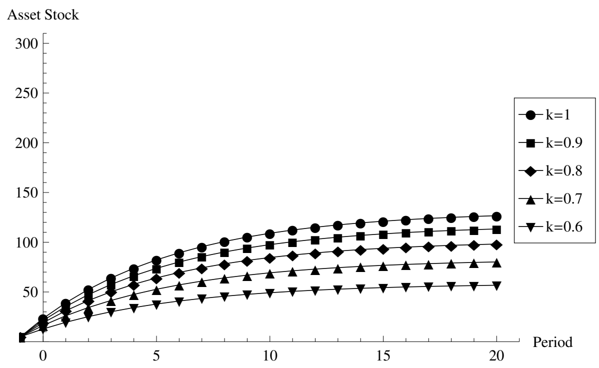
(a) Contestant i with $\eta = 0.5$



(b) Contestant i with $\eta = 1$



(c) Contestant j with $\eta = 0.5$

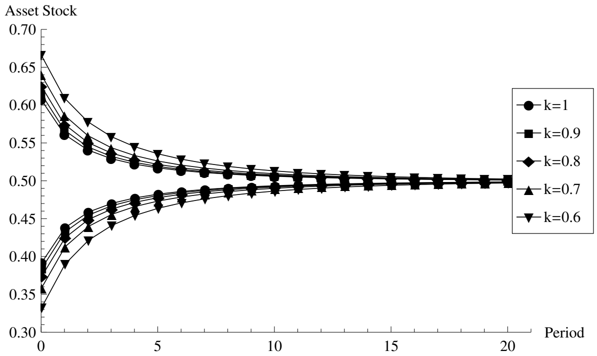


(d) Contestant j with $\eta = 1$

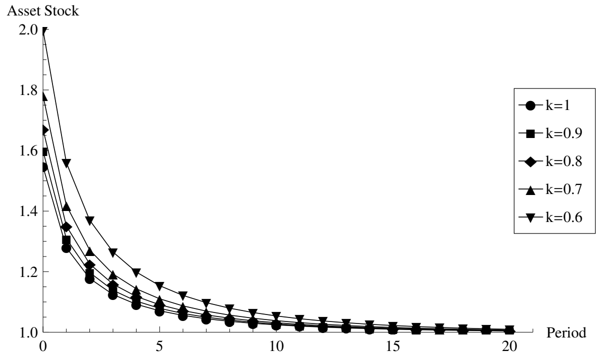
In Figures 1a and 1c, we set $\eta = 0.5$: that is, there is a relatively low elasticity of marginal costs. A higher k implies higher asset stocks during the transition as well as in the steady state. However, if we set $\eta = 1$ (see Figure 1b and 1d), that is, a higher elasticity of marginal costs, then the asset stock is lower in each period as compared to $\eta = 0.5$ for a given k. Nonetheless, a higher k implies a higher asset stock in each period as well.

We now turn our attention to how the prize spread affects the speed of convergence to the steady state. Figure 2, depicts the effect of a higher weight k attached to the first prize on the speed of convergence of the winning probabilities and hence on the balance of the contest. We use the same parameters as in Figure 1 and additionally set $\eta = 1$.

Figure 2: Second Prize Effect on Convergence of Winning Probabilities and Balance of Contest



(a) Convergence of $p_{i,t} > 0.5$ and $p_{j,t} < 0.5$



(b) Convergence of ratio $p_{i,t}/p_{j,t}$

In Figure 2a, a higher k implies a faster convergence of the winning probabilities, even if a higher weight k attached to the first prize implies a higher steady state asset stock. Figure 2b depicts this result: a higher parameter k increases the balance of the contest in each period during the transition. In the long run, however, contestants' winning probabilities are balanced. Note that a higher k increases asset stocks in each period for both contestants. However, increasing k has a relatively higher effect on the contestant with the lower initial asset stock.

We establish Corollary 3 which summarizes our key findings.

Corollary 3 *For a cost function with a constant elasticity of marginal costs, a higher weight k attached to the first prize will:*
(i) increase individual asset stocks as well as aggregate asset stocks during the transition as well as in the steady state itself,
(ii) produce a more balanced contest in each period as long as contestants start with different initial asset stocks,
(iii) induce a faster convergence to the steady state independent of the elasticity of marginal costs.

According to the corollary, there are three reasons for a contest designer to increase the prize spread between first and second prize in the case of a cost function with a constant elasticity of marginal costs:

First, if the contest designer aims to increase individual and aggregate asset stocks during the transition as well as in the steady state (see Section 2.2) itself, she/he should augment the spread between first and second prize because incentives to exert efforts increase in each period.

Second, a higher prize spread increases the balance of the contest in each period during the transition if the contestants start with different initial asset stocks.[12] In any case, a fully balanced contest is achieved in the long run (steady state) independent of the organizer's choice of the spread between first and second prize.

Third, a contest designer can increase the speed of achieving a balanced contest by increasing the weight attached to the first prize.

4 Conclusion

This paper has developed an infinitely repeated Tullock contest with a general cost function, in which two contestants contribute efforts to accumulate individual asset stocks. We extend the literature by analyzing the incentive effects of second prizes, which have not yet been analyzed in a dynamic contest model with a general cost function. In addition, we contribute to the literature by analyzing the effect of cost function specification on the speed of convergence of asset stocks. To investigate the transitional dynamics of the contest in the case of a general cost function, we use a linearized version of our model around the steady state. This linearization procedure, which has not yet been applied to a Tullock contest model, permits us to approximately determine the optimal path of asset stocks for both contestants.

Our analysis shows that in the long run (steady state), efforts and asset stocks increase with a higher prize spread and discount factor. On the other hand, a higher depreciation rate induces a decrease in steady state asset stocks but an increase in the steady state efforts. Our model further shows that optimal effort levels and their speed of convergence to the steady state depend on the stable Eigenvalue of the linearized system. In particular, the speed of convergence to the steady state is higher, the lower is the stable Eigenvalue. In the case of a cost function with a constant elasticity of marginal costs, a lower elasticity induces a faster convergence. Moreover, we find that the contestants' efforts monotonically increase over time into the steady state if initial assets stocks are larger than the steady state asset stocks. Moreover, a lower discount factor and/or a higher depreciation rate imply a lower Eigenvalue and therefore a higher speed of convergence. Our analysis further reveals that a higher spread between first and second prize increases aggregate asset stocks but does not alter the balance of the contest in the long run. During the transition, a higher prize spread increases the effort contributions of

[12] If contestants start with the same asset stocks, then the asset stocks of contestants are balanced in each period.

contestants as well as the balance of the contest in each period. Finally, a higher prize spread increases the speed of convergence to the steady state.

Our study can be seen as a first step to elucidate the transitional dynamics in an infinitely repeated Tullock contest with multiple prizes and a general cost function. We encourage further research in this area. For example, one promising avenue for further research might be the extension of our model to more than two contestants. Furthermore it would be interesting to see how our results carry over to a setting in which contestants are able to observe the opponents' effort levels after each period (closed-loop concept).

A Appendix

A.1 Derivation of the Euler equation, steady states and comparative statics

1. Euler equation:

Similar to Grossmann et al. (2010), we solve the dynamic program for contestant i and obtain:

$$v(E_{i,t-1}) = \max_{e_{i,t},E_{i,t}} \{p_i(E_{i,t}, E_{j,t})kV + (1 - p_i(E_{i,t}, E_{j,t}))(1 - k)V - C(e_{i,t}) + \beta v(E_{i,t})\}$$
$$\text{subject to } E_{i,t} = (1 - \delta)E_{i,t-1} + e_{i,t},$$

where $v(\cdot)$ represents the contestant's value function. Note that contestant i takes $E_{j,t}$ as given in period $t \in \{0, ..., \infty\}$ according to the open-loop concept. The Lagrangian L with multiplier $\lambda_t \in \mathbb{R}_0^+$ of the maximization problem is defined as:

$$L = \frac{kVE_{i,t}^{\gamma} + (1 - k)VE_{j,t}^{\gamma}}{E_{i,t}^{\gamma} + E_{j,t}^{\gamma}} - C(e_{i,t}) + \beta v(E_{i,t}) + \lambda_t[(1 - \delta)E_{i,t-1} + e_{i,t} - E_{i,t}].$$

Maximazing Ł with respect to $E_{i,t}, e_{it}$ and λ_t yields the following first-order conditions:

$$\begin{aligned} C'(e_{i,t}) &= \lambda_t, \\ \frac{kV\gamma E_{i,t}^{\gamma-1}E_{j,t}^{\gamma} - (1 - k)V\gamma E_{i,t}^{\gamma-1}E_{j,t}^{\gamma}}{\left(E_{i,t}^{\gamma} + E_{j,t}^{\gamma}\right)^2} + \beta\frac{\partial v(E_{i,t})}{\partial E_{i,t}} &= \lambda_t, \\ (1 - \delta)E_{i,t-1} + e_{i,t} &= E_{i,t}. \end{aligned}$$

Using the first-order conditions and the envelope theorem $\frac{\partial v(E_{i,t-1})}{\partial E_{i,t-1}} = \lambda_t(1-\delta)$, we obtain the following Euler equation for contestant i:

$$\frac{(2k - 1)V\gamma E_{i,t}^{\gamma-1}E_{j,t}^{\gamma}}{(E_{i,t}^{\gamma} + E_{j,t}^{\gamma})^2} = C'(e_{i,t}) - \beta(1 - \delta)C'(e_{i,t+1}).$$

2. Steady state:

We derive the following results:

(i) It is easy to show by a proof of contradiction that $E_i = E_j \equiv E$ in the steady state independently of initial asset stocks, if contestants have a strictly convex cost function. Furthermore, we obtain $e_i = e_j$ because $e_i = \delta E_i$ and $e_j = \delta E_j$.

(ii) Moreover, we find that a second prize has no effect on the balance of the contest in the long run. Note that $E_i = E_j$ holds independent of k. Hence,

$$\frac{E_i^{\gamma}}{E_i^{\gamma} + E_j^{\gamma}} = p_i(E_i, E_j) = \frac{1}{2} = p_j(E_i, E_j) = \frac{E_j^{\gamma}}{E_i^{\gamma} + E_j^{\gamma}},$$

which implies that the prize spread has no effect on the balance of the contest in the long run. Using the Euler equation and $E_i = E_j$, we conclude that $\frac{(2k-1)\gamma V}{4E} = [1-\beta(1-\delta)]\, C'(\delta E)$ implicitly defines E and therefore also $e = \delta E$.

3. Comparative statics:

To prove the comparative statics results, we use the implicit function theorem and obtain

$$\frac{\partial E}{\partial k} = -\frac{V\frac{\gamma}{2E}}{-(2k-1)V\frac{\gamma}{4E^2} - (1-\beta(1-\delta))C''(\delta E)\delta} > 0,$$

and

$$\frac{\partial e}{\partial k} = -\frac{V\frac{\delta\gamma}{2e}}{-(2k-1)V\frac{\delta\gamma}{4e^2} - (1-\beta(1-\delta))C''(e)} > 0.$$

Therefore, the steady state value E and e is increasing in k. Analogously, it is easy to show that $\partial E/\partial\beta > 0$, $\partial e/\partial\beta > 0$ and $\partial E/\partial\delta < 0$, $\partial e/\partial\delta > 0$.

A.2 Proof of Lemma 1

To investigate whether $|\mu_1| < 1, |\mu_2| < 1, |\mu_3| > 1$ and $|\mu_4| > 1$ actually emerge as such, we compute the Eigenvalues of Q, based on the characteristics of the matrix Q, as follows:

$$\begin{aligned}\mu_{1,2} &= \frac{\frac{\gamma V(2k-1)+4E^2(1+\beta(1-\delta)^2)C''(e)}{4E^2\beta(1-\delta)C''(e)} - \sqrt{\left(\frac{\gamma V(2k-1)+4E^2(1+\beta(1-\delta)^2)C''(e)}{4E^2\beta(1-\delta)C''(e)}\right)^2 - \frac{4}{\beta}}}{2}\\ &= \frac{\frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)} - \sqrt{\left(\frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)}\right)^2 - \frac{4}{\beta}}}{2}\end{aligned}$$

$$\begin{aligned}\mu_{3,4} &= \frac{\frac{\gamma V(2k-1)+4E^2(1+\beta(1-\delta)^2)C''(e)}{4E^2\beta(1-\delta)C''(e)} + \sqrt{\left(\frac{\gamma V(2k-1)+4E^2(1+\beta(1-\delta)^2)C''(e)}{4E^2\beta(1-\delta)C''(e)}\right)^2 - \frac{4}{\beta}}}{2}\\ &= \frac{\frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)} + \sqrt{\left(\frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)}\right)^2 - \frac{4}{\beta}}}{2}\end{aligned}$$

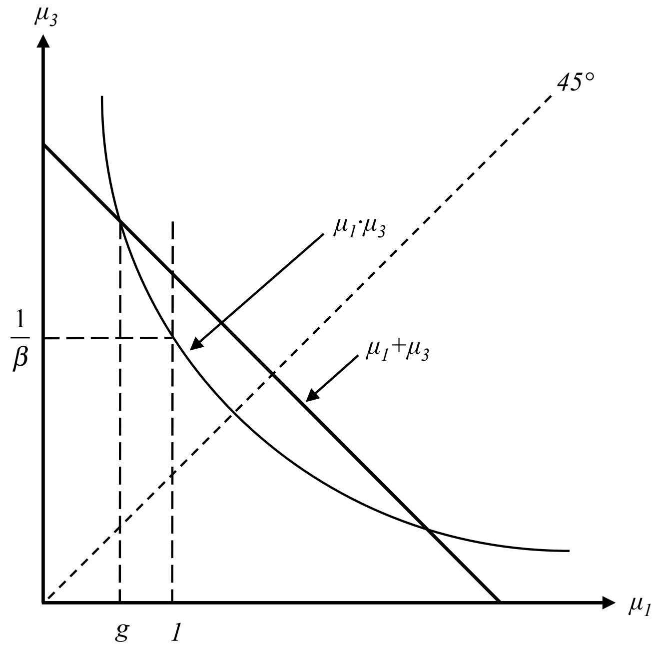
Hence, we conclude that there are just two different Eigenvalues $\mu_1 = \mu_2$ and $\mu_3 = \mu_4$. The sum and the product of the two Eigenvalues are as follows:

$$\begin{aligned}\mu_1 + \mu_3 &= \mu_2 + \mu_4 = \frac{\gamma V(2k-1) + 4E^2(1+\beta(1-\delta)^2)C''(e)}{4E^2\beta(1-\delta)C''(e)}\\ &= \frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)} \qquad (9)\end{aligned}$$

$$\mu_1 \cdot \mu_3 = \mu_2 \cdot \mu_4 = \frac{1}{\beta} \qquad (10)$$

It is easy to see that $\mu_1 + \mu_2 > 0$ and $\mu_1 \cdot \mu_2 > 0$. Hence, equation (9) together with equation (10) imply that μ_1 and μ_3 are both positive roots. We use this intermediate result further below. Figure 3 graphically illustrates equations (9) and (10).

Figure 3: Stable and Unstable Roots



In system (7), there are two predetermined variable $(E_{i,t}, E_{j,t})$ and two non-predetermined variables $(e_{i,t}, e_{j,t})$. Blanchard and Kahn (1980) have shown that if the number of non-predetermined variables equals the number of roots outside the unit circle, then there exists a unique solution. Next, we provide a proof that there is indeed a unique solution in system (7). Without loss of generality, we assume that $\mu_3 > \mu_1$. Suppose that $\mu_1 = 1$. It follows from equation (10) that $\mu_3 = 1/\beta > 1$. Furthermore, $\mu_1 + \mu_3 = (1+\beta)/\beta$. However, we also know from equation (9) that $\mu_1 + \mu_3 = \delta(1-\beta(1-\delta))/(\sigma(e)\beta(1-\delta)) + \left(1+\beta(1-\delta)^2\right)/(\beta(1-\delta))$. Hence, we suggest by inspecting Figure 3 that $\mu_1 < 1$ (and therefore $\mu_3 > 1$) if and only if the following condition is satisfied:[13]

$$\frac{1+\beta}{\beta} < \frac{\delta(1-\beta(1-\delta))}{\sigma\beta(1-\delta)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)} \Longleftrightarrow 1 > -\sigma \tag{11}$$

Initially, we have assumed a strictly convex cost function that implies that the inequality condition (11) is always fulfilled. Therefore, we have two stable roots $(\mu_1 = \mu_2)$ and two unstable roots $(\mu_3 = \mu_4)$ such that there is indeed a unique solution. In addition, we have shown that uniqueness is assured for a set of concave cost functions as long as the

[13]Note that the curve $\mu_1\mu_3$ converges to the axes. Therefore, the upper intersection of the two curves lies to the left side of $\mu_1 = 1$ such that $(1-g) > 0$ with $g < 1$.

function is not "too" concave, i.e., such that $1 > -\sigma$ is still satisfied. This completes the proof.

A.3 Proof of Proposition 1

We transform the second-order difference equation (8) into the following form using new variables u, v and w.

$$\begin{aligned} 0 &= [(1-uL)(1-vL^{-1})w]\hat{E}_{i,t+1} \\ 0 &= [w - uwL - vwL^{-1} + uvw]\hat{E}_{i,t+1} \end{aligned} \tag{12}$$

Comparing equation (8) with equation (12) by using the method of undetermined coefficients yields three restrictions for u, v and w:

$$\text{(i) } w + uvw = -(a+f), \text{ (ii) } -uw = (af - bm), \text{ (iii) } -vw = 1$$

The combination of (i), (ii) and (iii) allows us to write:

$$\begin{aligned} \frac{(bm-af)}{u} - u &= -(a+f) \\ (bm-af) - u^2 &= -(a+f)u \\ u^2 - (a+f)u + (af-bm) &= 0 \end{aligned}$$

We conclude that there are two solutions for u:

$$u_{1,2} = \frac{\frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)} \pm \sqrt{(\frac{\delta(1-\beta(1-\delta))}{\beta(1-\delta)\sigma(e)} + \frac{1+\beta(1-\delta)^2}{\beta(1-\delta)})^2 - \frac{4}{\beta}}}{2}$$

It is easy to see that $u_{1,2}$ equals $\mu_{1,3}$ or $\mu_{2,4}$. Henceforth, we interchangeably use both notations. Note that μ_1 corresponds to the stable solution. Therefore, u_1 is also a stable root. Using the last equation and rewriting equation (12), we obtain:

$$0 = [(1-u_1L)(1-vL^{-1})w]\hat{E}_{i,t+1}$$

$$\begin{aligned} &\iff [(1-u_1L)(1-vL^{-1})w]E_{i,t+1} = [(1-u_1L)(1-vL^{-1})w]E \\ &\iff (E_{i,t+1}-E) - u_1(E_{i,t}-E) = v(E_{i,t+2}-E) - vu_1(E_{i,t+1}-E) \end{aligned} \tag{13}$$

Next, we utilize the method of undetermined coefficients again. Suppose that the functional form of the solution looks like $E_{i,t+1} - E = \zeta(E_{i,t} - E)$. This is the guess. Now, we substitute this guess in equation (13) to determine the coefficient ζ:

$$\begin{aligned}(E_{i,t+1}-E)-u_1(E_{i,t}-E) &= v(E_{i,t+2}-E)-vu_1(E_{i,t+1}-E)\\ \zeta-u_1 &= v\zeta(\zeta-u_1).\end{aligned}$$

As long as $v\zeta \neq 0$, we obtain the restriction $\zeta = u_1$ for the last equation. Hence, we conclude that

$$E_{i,t+1}-E=\zeta(E_{i,t}-E)=u_1(E_{i,t}-E).$$

Next, we replace u_1 with μ_1 such that $E_{i,t+1}-E=\mu_1(E_{i,t}-E)$. Recursively, we find that: $E_{i,t}-E=\mu_1^{t+1}(E_{i,-1}-E)$. We notice that the dynamics of the asset stock crucially depend on the stable Eigenvalue $\mu_s \equiv \mu_1$ with $\mu_1 < 1$.[14] A high value of μ_s implies that the asset stock slowly converges to the steady state and vice versa. Next, we consider the optimal policy function e_t. From the linearization around the steady state, we know that

$$\begin{aligned}\hat{E}_{i,t+1} &= (1-\delta)\hat{E}_{i,t}+\delta\hat{e}_{i,t+1}\\ &\iff E_{i,t+1}-E=(1-\delta)\left(E_{i,t}-E\right)+\left(e_{i,t+1}-e\right)\end{aligned}$$

Earlier, we have computed that $E_{i,t+1}-E=\mu_s(E_{i,t}-E)$. Using the last two equations, we obtain:

$$\begin{aligned}\mu_s(E_{i,t}-E) &= (1-\delta)\left(E_{i,t}-E\right)+\left(e_{i,t+1}-e\right)\\ &\iff e_{i,t+1}=e+(\mu_s+\delta-1)\mu_s^{t+1}(E_{i,-1}-E)\\ &\iff e_{i,t}=e+(\mu_s+\delta-1)(E_{i,-1}-E)\mu_s^t\end{aligned}$$

Using symmetry, we obtain the following dynamics for efforts and asset stocks after linearizing the model around the steady state:

$$\begin{aligned}e_{i,t} &= e+(\mu_s+\delta-1)(E_{i,-1}-E)\mu_s^t\\ e_{j,t} &= e+(\mu_s+\delta-1)(E_{j,-1}-E)\mu_s^t\\ E_{i,t} &= E+(E_{i,-1}-E)\mu_s^{t+1}\\ E_{j,t} &= E+(E_{j,-1}-E)\mu_s^{t+1}\end{aligned}$$

with $i,j\in\{1,2\}$ and $j\neq i$. The last four equations exactly represent Proposition 1. This completes the proof.

[14]Note that $\mu_s > 0$, as shown in the proof of Lemma 1.

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