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Aggregative Contests and Ex-post Heterogeneity: the Case of the UEFA Champions League

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Abstract

The UEFA Champions League is an annual Pan-European football competition that takes place parallel to the domestic league competitions. The participation in the Champions League secures the teams large payments, which have steadily increased over the last decade. This paper develops a general contest model of a professional team sports league in which the winner of the domestic league competition participates in the Champions League and has the possibility to win an additional prize. Our model considers two leagues where each league is composed out of several profit-maximizing clubs. We analyze the impact of the Champions League on investment incentives, competitive balance and club profits. Our model shows that the Champions League prize and domestic league prize have differential effects on competitive balance: an increase in the Champions League prize decreases competitive balance, while an increase in the domestic league prize increases competitive balance.

Keywords: Aggregative game, contest, heterogeneity, team sports league

JEL Classification: C72, D43, D72, L13, L83
1 Introduction

In contrast to their American counterparts, football leagues in Europe are embedded in association structures. Every national football association governs a system of leagues that is open through promotion and relegation from the amateur level to the top national division of professional football. At the top of the national league pyramid, the UEFA (Union of European Football Associations), an association of national associations, organizes European club competitions like the Champions League and the Europa League for teams meeting certain sportive qualification criteria. The Champions League and the Europa League take place at the same time as the domestic league competitions and secure the participating teams large additional revenues. The format of the Champions League has changed several times in the past. The major modification was probably the extension of the participating teams from 24 to 32 in the 1999-2000 season combined with a change in the payout structure.

According to Pawlowski et al. (2010), the payments to the participating teams in the Champions League have experienced a large increase since this modification in the 1999-2000 season. Simultaneously to the increased value of Champions League participation, Pawlowski et al. (2010) find empirical evidence that competitive balance in the Big Five European football leagues has experienced a ”persistent decline”.\footnote{The ”Big Five” leagues in Europe are: Premier League (England, 20 clubs), Ligue 1 (France, 20 clubs), Bundesliga (Germany, 18 clubs), Primera Division (Spain, 20 clubs) and the Serie A (Italy, 18 clubs).} The authors attribute this decline in competitive balance to the significant increase in the Champions League payments to the participating teams. However, the authors also acknowledge that the domestic championships in the Big Five leagues have simultaneously augmented in value due to increased ”domestic media revenues, investor market entries, and successful internationalization efforts by European football clubs.” This consideration raises the question: Is there a different effect of an increase in the Champions League prize and domestic league prize on competitive balance?\footnote{According to the so-called “uncertainty of outcome hypothesis,” fans prefer to attend games with an uncertain outcome ("competitive balance") and enjoy close championship races (Rottenberg 1956 and Neale 1964). For theoretical studies that deal with competitive balance in team sports leagues, see e.g., Kéenne (2000b), Szymanski & Kéenne (2004), Grossmann et al. (2010) and Lang et al. (2011).}

In this paper, we propose a general theory that allows to answer this and related questions. Particularly, we develop a general framework of a contest model that can be
applied to a professional team sports league in which the winner of the domestic league competition participates in the Champions League and thus it has the possibility to win an additional prize. We analyze how changes in the Champions League and domestic league prizes affect talent investments, win percentages and club profits for so-called ”aggregative games” (Corchon 1994). In such aggregative games the payoffs depend only on individual talent investments and an aggregate of all talent investments. Hence, to derive our results, we assume ”aggregate taking behavior” (ATB), i.e., the clubs choose their talent investments by taking the aggregate talent investments in the league as given. We also provide robustness tests for the results derived in our general framework applying numerical simulations. For example, we show that our results are robust with respect to the assumption of aggregate taking behavior because we obtain qualitatively similar results under the classical Nash concept.

Before we proceed with the model, we briefly review the related literature. Our paper is naturally related to the literature on sports economics. Analytical models in sports are mainly focused on the effect of cross-subsidization schemes such as revenue sharing and salary caps on competitive balance in one-stage contest models. The recent sports economics literature has suggested modeling a team sports league by making use of contest theory. In his seminal article, Szymanski (2003) applied Tullock’s (1980) rent-seeking contest to ascertain the optimal design of sports leagues. However, to the best of our knowledge, there is no contest model in sports that analyzes the effect of Champions League and domestic league prize variations on ex-post heterogeneity. Besides the contribution to the sports literature, our paper relates also to the contest literature, particularly, to the literature on imperfectly discriminating contests. It is a well-known
result in one-stage contests that variations in the contest prize have no effect on relative effort levels and profits (Tullock 1980). We contribute to the literature on one-stage contests by generalizing the results of prize variations on ex-post heterogeneity to a very general framework of a contest model. Moreover, our paper is related also to the literature on multi-stage contests, in particular, multi-stage sequential-elimination contests. In this area, the most well-known structure is probably the multi-stage pairwise elimination contest ("knock-out tournament"). See e.g., Gradstein & Konrad (1999), Stein & Rapoport (2004), Harbaugh & Klumpp (2005), Klumpp & Polborn (2006) and Fu & Lu (2012).

The remainder of the paper is organized as follows. In the next section, we introduce our general model framework. In Section 3, we analyze isolated leagues with exogenous and endogenous league revenues. In Section 4, we then investigate leagues which are integrated in a superior Champions League: First we analyze across-league symmetry and within-league heterogeneity. Second, we investigate across-league heterogeneity and within-league symmetry. Section 5 provides robustness checks via numerical simulations. Finally, Section 6 discusses the results and concludes the paper.

2 Model

2.1 General Setup

Let \( I = [0, 1] \) and \( B \) the Borel set over \( I \) and \( \mu \) is a measure over \((I, B)\). \( i \in I \) simply is the identity of a club in a league. To start, we consider the simple expected payoff-function of club \( i \):

\[
\Pi(i) = p(i)R - C(i)
\]

(1)

where \( R > 0 \) is a constant contest prize, i.e., the championship prize, and \( p(i) \) is a non-negative \( \mu \)-integrable probability function with \( \int_{0}^{1} p(s)\mu(ds) = 1 \). Thus, the probabilities to win the contest sum up to one in the league. By choosing the appropriate measure \( \mu \), this setting encompasses the case of discrete (or countable) clubs as well as the case of

\[9\] For multi-stage contests that analyze perfectly discriminating contests (so-called "all pay auctions"), see e.g., Moldovanu & Sela (2006) and Groh et al. (2012).

\[10\] We can think of (1) in the context of a general contest (not necessarily a sports contest) as follows: For a given distribution of \( p(i) \), (1) depicts the expected payoff from a winner-takes-it-all contest, where \( p(i) \) is the probability of player \( i \) winning the contest, or as the effective payoff that \( i \) gets from holding a market share of \( p(i) \).
of "zero-mass"-clubs. In the former case we set $\int \mu(ds) = n \geq 2$, whereas in the latter case $\mu$ simply is the Lebesgue measure, i.e. $\int \mu(ds) = 1$. In any case, $C(i)$ is the costs associated with club $i$.

Suppose that $p(i)$ depends on club $i$'s individual effort $t(i)$ and some aggregate effort:

$$p(i) = P \left( f(t(i)), \int f(t(s))\mu(ds) \right)$$

where $f$ and $t$ both are integrable non-negative functions, with $f' > 0$, $f'' \leq 0$, and $P \in C^2(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ with $P_1 > 0$ and $P_{11} \leq 0$. More specifically, we assume that $f$ is a power function with exponent $\theta \in (0, 1]$.

Further, let

$$C(i) = c(i)h(t(i))$$

where $c(i) \in [1, \bar{c}]$ is an integrable function and $h \in C^2(\mathbb{R}^+, \mathbb{R}^+)$ is strictly increasing and strictly convex function. To simplify matters, we assume that $h$ is a power function $h(z) = z^\eta$ with $\eta > 1$ which implies that this function is homogeneous and multiplicatively separable ($h(xy) = u(x)v(y)$).

All clubs simultaneously and non-cooperatively maximize their payoff-function (1). Hence they solve:

$$\max_{t(i)} P \left( f(t(i)), \int f(t(s))\mu(ds) \right) R - c(i)h(t(i))$$

s.t. $\int P \left( f(t(z)), \int f(t(s))\mu(ds) \right) \mu(dz) = 1$ (2)

**Example:** A special case is given by the Tullock success function with $f(x) = x^r$. Then:

$$P \left( f(t(i)), \int f(t(s))\mu(ds) \right) = \frac{t(i)^r}{\int t(s)^r\mu(ds)}$$

As we are mainly interested in the equilibrium distribution of $p(i)$ (rather than $t(i)$) we can use the injectivity of the cost function to work with a version where the clubs choose $p(i)$ directly. The main advantage is that how strongly a club might affect his winning chance $p(i)$ by changing his effort can be captured by the cost function.

\[\text{In our context, effort means clubs’ (financial) investments in players, coaches, staff, infrastructure and so forth.}\]
To see this let $e(i) = f(t(i))$. Then $t(i) = f^{-1}(e(i))$. Hence

$$\Pi(i) = P(e(i), \int e(s)\mu(ds) R - c(i)k(e(i))$$

where $k(e(i)) \equiv h(f^{-1}(e(i)))$. Note that the assumptions on $f$ and $h$ imply that $k$ is a strictly increasing and strictly convex function.

Finally, we assume that $P(t, T)$ is a zero-homogenous function\(^{12}\).

**Proposition 1** Suppose that the function $P(t, T)$ is zero-homogenous. Then payoffs can be written as

$$\Pi(i) = p(i)R - c(i)k(T)k(\varphi(p(i)))$$

where $T = \int e(s)\mu(ds)$ and $\varphi \in C^2$ with $\varphi' > 0$ and $\varphi'' \geq 0$.

**Proof** See Appendix.

### 2.2 Aggregate Taking Behavior

The one additional assumption we make compared to standard contest theory is that we assume that the clubs choose their strategy taking the aggregate $T$ as given. We use this assumption because of three reasons:

- First, if many clubs are involved, it is plausible that the competitors have an idea of some aggregate (or average) effort rather than what individual participants do.

- Second, this approach leads to high analytical tractability - especially when it comes to examining how ex-post heterogeneity of success and profits depend on the parameters of the model.

- Third, an aggregate taking equilibrium (ATE) is a reasonable approximation to the Nash equilibrium (NE).

To illustrate this last point, consider the following specific profit function:

$$\Pi(i) = p(i)R - c(i)T^2p(i)^2$$

\(^{12}\)Note that any Tullock success function satisfies this property.
Suppose that the power of $f$ in $p(i)$ is 1 and there is a finite number of clubs $n$, i.e. (by relabelling club’s identity) $T = \int t(s)\mu(ds) = \sum_{i=1}^{n} t(i)$.

The conventional best-response (BR) of club $i$ is found by solving

$$\max_{p(i)} p(i)R - \frac{1}{2}c(i)T^2p(i)^2 \quad s.t. \quad T = \sum_{i=1}^{n} t(i).$$

Under aggregate taking behavior, clubs ignore their effect for the aggregate and they solve

$$\max_{p(i)} p(i)R - \frac{1}{2}c(i)T^2p(i)^2.$$  

The difference is that for the BR the clubs take into account their direct effect on $T$ when they optimize whereas with ATB they do not. In both cases $T = \sum_{i=1}^{n} t(i)$ holds in equilibrium.

For the case of BR we get as equilibrium conditions

$$p_{BR}(i) = \frac{R}{c(i)T_{BR}^2 + R} \quad \text{and} \quad 1 = \sum_{i=1}^{n} \frac{R}{c(i)T_{BR}^2 + R}$$

and for ATB we get

$$p_{ATB}(i) = \frac{R}{c(i)T_{ATB}^2} \quad \text{and} \quad 1 = \sum_{i=1}^{n} \frac{R}{c(i)T_{ATB}^2}$$

The two solutions can be easily compared in the case of a symmetric league (i.e. $c(i) = c \ \forall i$). Then $p_{ATB}(i) = p_{BR}(i) = 1/n$ and

$$T_{BR} = \sqrt{\frac{R}{c(n-1)}} \quad T_{ATB} = \sqrt{\frac{R}{cn}} = T_{BR}\sqrt{\frac{n}{n-1}} > T_{BR}$$

Thus, we see that $T_{ATB}$ is larger than $T_{BR}$ by a factor, which decreases to 1 as $n$ gets large. Intuitively, we get $T_{ATB} > T_{BR}$ because under BR the clubs take into account that, if they increase their effort, this has the direct effect of increasing their winning chance but at the same time there is a countervailing indirect effect as $T$ increases. This indirect effect is not taken into account under ATB which explains the difference in aggregate effort (as well as in individual effort levels $t(i)$).
In the following we will predominantly concentrate on the case of ATB. However, we check the robustness of our main ATB results applying simulations in section 5 where we use the classical Nash equilibrium.

3 Isolated Leagues

3.1 Exogenous Prize

In alignment with the above considerations we consider payoff-functions of the form

\[ \Pi(i) = p(i)R - c(i)K(T)H(p(i)) \]

with \( R \) as the exogenous championship prize. \( K' > 0 \) and \( K'' > 0 \), while \( H(\cdot) \) is a power function (with exponent larger than 1), and \( H(0) = K(0) = 0 \). Ex-ante heterogeneity is given by the distribution of \( c(i) \) over clubs. In detail: \( c \in C^1(I, [1, \bar{c}]) \) with \( c'(i) \geq 0 \).

The equilibrium conditions are given by

\[ R = c(i)K(T)H'(p(i)) \quad \int p(s)\mu(ds) = 1 \quad (4) \]

Theorem 1 There exists a unique function \( p : I \rightarrow (0, 1) \) that solves (4) and the equilibrium aggregate \( T \) is uniquely determined. Moreover, \( p(i) \) is a weakly decreasing and differentiable function.

Proof See Appendix.

Remark: In the case where \( \mu \) is a continuous measure (i.e. where clubs have zero-mass), \( p(i) \) is a (strictly positive) density. If \( \mu \) is discrete, then evaluating \( p(i) \) at points of unit mass of \( \mu \) gives the probability distribution \( p(i) \) over the clubs.

Corollary 1 If \( c(i) \) is strictly increasing. Then, \( p(i) \) and \( \Pi(i) \) are strictly decreasing functions in equilibrium.

Proof See Appendix.

Remark: This is what can be intuitively expected: ex-ante cost advantages (i.e., a lower \( i \), since \( c(i) \) is increasingly ordered in \( i \)) translate into higher ex-post winning
probabilities and expected profits.

How does a change of $R$ affect the distribution of $p(i)$ and $\Pi(i)$? From (22) in the appendix, we see that $p(i)$ is independent of $R$.\footnote{This result is driven by the assumption of homogeneous costs (because then (22) is independent of $T$) but not by aggregate taking behavior.} Some algebra then yields the following result:

**Proposition 2** In equilibrium, all clubs earn positive expected profit. The winning probability function $p(i)$ is independent of $R$ and so are relative winning chances $p(i)/p(j)$. The equilibrium profit function $\Pi(i)$ and $T$ depend strictly positively on $R$ but relative profits $\Pi(i)/\Pi(j)$ are independent of $R$.

**Proof** See Appendix.

**Conjecture:** The level of expected overall profits never exceeds $R$. (This is easy to see when $\mu$ is a point measure but harder in the continuous case.)

**Remark:** The result shows that, with homogeneous costs, an exogenous change of the prize never has an effect on the winning distribution nor on relative profits. Intuitively, this occurs as doubling the prize exactly leads to a doubling of the individual costs. Hence, when looking at relative profits, these effects cancel out. However, as $\Pi(i)$ is an increasing function of $R$, the result also implies that the absolute difference in profits, $\Pi(i) - \Pi(j)$, $i < j$, increases as $R$ increases.

The result further shows that $R$ is an ineffective instrument to change the distribution of ex-post winning probabilities in the simple fixed-prize contest setting with homogeneous costs (e.g. in the Tullock world).

### 3.2 Endogenous Prize

In this section we consider $R = R(\psi, T)$ with $R \in C^2((0, \infty)^2, (0, \infty))$ and $R_1 > 0$. $\psi > 0$ is an exogenous prize shifter. In our context, $\psi$ can be interpreted as the championship prize in the league. The overall prize $R$ is endogenous in this subsection, since aggregate effort influences the size of the prize. For instance, more club effort can increase the league’s attractiveness such that the broadcasting revenues increase in the league which implies that the size of the championship prize increases. The expected profit function is

$$\Pi(i) = p(i)R(\psi, T) - c(i)K(T)H(p(i))$$
Further, the following conditions are helpful:

\[
\lim_{T \to 0} \frac{R(\psi, T)}{K(T)} = \infty \quad \lim_{T \to \infty} \frac{R(\psi, T)}{K(T)} = 0
\]  

(5)

\[
\frac{R_T}{R} < \frac{K_T}{K} \quad \forall \psi, T > 0
\]  

(6)

The equilibrium conditions are

\[
R(\psi, T) - c(i)K(T)H'(p(i)) = 0 \quad \int p(s)\mu(ds) = 1
\]  

(7)

**Proposition 3** Suppose condition (5) holds. Then there exists a unique function \( p : I \to (0, \infty) \) and a positive number \( T \) that solve (7), and \( p(i) \) is a strictly decreasing and differentiable function. If additionally condition (6) holds, the aggregate \( T \) is uniquely determined and a \( C^1 \)-function of \( \psi \).

**Proof** See Appendix.

**Remark** If \( \mu \) is the discrete measure, then \( p(i) \in (0, 1) \). If \( \mu \) is continuous, then \( p(i) \) is a density (hence \( p(i) \geq 1 \) is possible). As \( p(i) \) is independent of \( \psi \), corollary 1 applies again.

Regarding the comparative statics with respect to \( \psi \) some caution must be paid if \( R_T > 0 \). If the function \( R \) reacts strongly positive to a change of \( T \) and \( T \) is slightly larger than the equilibrium value \( T^* \), this may induce clubs to increase their effort, which increases \( T \) and so on - so we would not reach \( T^* \) in the end. To avoid the divergence of such a dynamic we adopt the concept of gradient dynamics to the aggregative game.

Suppose \( p_i \equiv p(i) \) is a time-dependent function \( p_i(\tau) \), where the adjustments, \( \dot{p}_i(\tau) \), are proportional to \( \partial \Pi(i)/\partial p(i) \):

\[
\dot{p}_i(\tau) = s \left( R(\psi, T) - c(i)K(T)H'(p(i)) \right) \quad s > 0
\]  

(8)

Consider a small deviation from equilibrium: \( p_i = p_i^* + \varepsilon \). When does the dynamic given by (8) lead to \( \lim_{\tau \to \infty} p_i(\tau) = p_i^* \)? Using\(^{14} \) \( T'(p_i) = T/(1 - p_i) \), this holds if

\[
\frac{R_T}{R} \left( \frac{K'(T^*)}{K(T^*)} \right) < \frac{H''(p_i^*)}{H'(p_i^*)} \frac{1 - p_i^*}{T^*}
\]

\(^{14}\)Strictly spoken, this assumes that \( \mu \) is the discrete measure.
Hence we may conclude that if the equilibrium aggregate $T^*$ is uniquely determined, then (8) is locally stable - and comparative statics may be found in the usual way. Henceforth, we assume conditions (5) - (6) to hold.

**Corollary 2** In equilibrium, $p(i)$ and $\Pi(i)$ are strictly decreasing functions of $i$. Neither $p(i)$ nor relative profits $\Pi(i)/\Pi(j)$ depend on $\psi$. However, the aggregate $T$ and the level of profits $\Pi(i)$ are strictly increasing in $\psi$.

**Proof** See Appendix.

**Remark:** If $R$ is endogenously determined only by the aggregative quantity $T$, then relative ex-post heterogeneity is unaffected by exogenous shifts of the prize. This result will change (in the next section), if $R$ also depends directly on the winning probability $p(i)$. Also, the result depends on the assumption of ATB.$^{15}$

## 4 Integrated Leagues

In this section we analyze contests with prize function $R = R(\Psi, T, p(i))$. $\Psi = (\psi, \Omega)$ is a prize vector with two elements. The first element $\psi$ is a prize shifter and stands for the championship prize in the home league similar to the last section. In addition, $\Omega$ represents the prize in the superior league like the Champions League in the European soccer league.

According to the formulation, the prize $R$ depends on the winning probability. When a European soccer club wins the championship in the home league, this club has the opportunity to participate in the Champions League. Therefore, higher effort not only implies a higher chance to win the championship prize $\psi$ in the home league, but it also increases the chance to win the Champions League prize $\Omega$. Therefore, we capture the Champions League effect by introducing the winning probability and $\Omega$ as an argument in the prize function $R$.

In full generality, the model is analytically not sufficiently tractable to describe general comparative-static effects of $\psi$ and $\Omega$ on the ex-post distribution of $p(i)$. Nevertheless, some basic results can be obtained and are presented next. Afterwards, we simplify by letting $R$ be a linear function of some shift parameters.

$^{15}$E.g. in the conventional two-player Tullock-model with $R = \psi + (t_1 + t_2)$ the relative chance $p_1/p_2$ is a decreasing function of $\psi$. 

11
Consider
\[
\Pi(i) = p(i) R(\Psi, T, p(i)) - c(i) K(T) H(p(i))
\]
with \(H\) as a power function with exponent not less than 2. We henceforth assume that \(R_p > 0, R_{pp} \leq 0\) and \(R_p \leq R_{p(i)}\), which asserts that
\[
\frac{\partial \Pi(i)}{\partial p(i)} = 0 \Rightarrow \frac{\partial^2 \Pi(i)}{\partial p(i)^2} < 0
\]

Then, if a solution to \(\frac{\partial \Pi(i)}{\partial p(i)} = 0\) exists, it is unique. Note that, for given \(T\) and \(i\), \(p(i)\) is found. The equilibrium conditions are
\[
R(\Psi, T, p(i)) + p(i) R_p(\Psi, T, p(i)) = c(i) K(T) H'(p(i)) \int p(i) \mu(di) = 1 \quad \forall i \quad (9)
\]

Again, an equilibrium is a differentiable function \(p(i)\) and an aggregate \(T\) such that (9) holds.

**Proposition 4 (No leap-frogging)** Any equilibrium \((p(i), T)\) of (9) has the property that \(p'(i), \Pi'(i) < 0\).

**Proof** See Appendix.

In the following we derive the Champions League model with the two phases more specifically. There are two leagues \(A\) and \(B\). In the first phase only clubs within a league compete against each other for a fixed league prize \(\psi^j > 0\), where \(j = A, B\). In the second phase the winners of league \(A\) and \(B\) compete against each other for a fixed Champions League prize \(\Omega > 0\). The important feature of our analysis is that initial efforts crucially determine both the chance to win phase one and two. This is the case e.g. where efforts correspond to strategic precommitment.

Hence the payoff of club \(i\) in league \(A\) has the form
\[
\Pi^A(i) = P^A_i(W_1) (\psi^A + P^A_i(W_2|W_1) \Omega) - C^A_i
\]  

where \(P^A_i(W_1)\) is the probability that club \(i\) wins in the home league (phase 1) and \(P^A_i(W_2|W_1)\) is the conditional probability that club \(i\) wins the Champions League (phase 2) given that club \(i\) won the home league (phase 1).

In the following we assume that both leagues have the same measure \(\mu\) of clubs and
the within league success function \( P(t, T) \) is formally the same for both leagues. Hence \( \int \mu(di) = n \) where \( n \in \mathbb{N} \) if \( \mu \) is discrete and \( n = 1 \) if \( \mu \) is continuous.

Leagues may however differ in the within-league prize \( \psi \) and the ex-ante distribution of costs \( c(i) \). Clubs take league aggregates \( T^A \) and \( T^B \) as given.

We now derive an expression for \( P^A_i(W_2|W_1) \). First, suppose that the first-phase success function is given by \( P(t^A(i), T^A) \) as before, where \( T^A = \int t^A(i) \mu(di) \). Suppose that club \( s \) of league \( B \) won the first-phase contest in league \( B \). If the final contest success function is similar to the within-league success function, the probability of club \( i \) of league \( A \) winning against \( s \) simply is \( P\left(t^A(i), \tilde{T}(s)\right) \), where \( \tilde{T}(s) = t^A(i) + t^B(s) \).

Then, the conditional probability that club \( i \) of league \( A \) wins the second-phase contest is

\[
P^A_i(W_2|W_1) = \int p^B(s) P\left(t(i), \tilde{T}(s)\right) \mu(ds)
\]

and (10) becomes

\[
\Pi^j(i) = P(t^j(i), T^j) \left( \psi^j + \Omega \int p^{-j}(s) P\left(t^j(i), \tilde{T}(s)\right) \mu(ds) \right) - c^j(i)h\left(t^j(i)\right)
\]

where \( j = A, B \). An aggregate taking equilibrium are two functions \( t^A(i), t^B(i) : \mathcal{I} \to \mathbb{R}^+ \) and aggregate quantities \( T^A, T^B \) such that \( (t^A(i), t^B(i), T^A, T^B) \) maximize (11) for any given \((i, j)\)-combination and \( T^j = \int t^j(i) \mu(di) \).

As before, we are interested in how changes of \( \psi^j \) or \( \Omega \) affect the equilibrium effort functions \( t^j(i) \) and the associated winning probabilities \( p^j(i) \) and profit levels \( \Pi^j(i) \). As there are two different levels of ex-ante heterogeneity, namely within-league heterogeneity versus across-league heterogeneity, potential effects are very hard to uncover without pursuing a rigid analysis. Therefore, we consider both the case of across-league symmetry but within-league heterogeneity and across-league heterogeneity but within-league symmetry separately. It is our understanding that only along such lines the comparative-statics of the model may be reasonably investigated.
4.1 Across-League Symmetry and Within-League Heterogeneity

In the following we assume that both leagues have the same cost distribution over the clubs, i.e. $c^A(i) = c^B(i) = c(i) \forall i$ and $\psi^A = \psi^B = \psi$. We consider only equilibria of the type $T^A = T^B = T$, that is we consider quasi-symmetric equilibria.

To allow for a tractable analysis we work with two further simplifications. First, we assume that the success function is given by the Tullock function, i.e.

$$P(t(i), T) = \frac{t(i)^r}{T} \tag{12}$$

with $T \equiv \int t(i)^r \mu(di)$.

Next, we assume that the cost distribution of league $B$ is highly concentrated about its mean, i.e. $c(i), c(j)$ are arbitrarily close to each other. Then $p^B(i) \approx 1/n$ and $t^B(i) \approx \left(\frac{T_B}{n}\right)^{1/r} \forall i$. In this case we have

$$\int p^B(s)P(t(i), \tilde{T}(s)) \mu(ds) \approx \frac{(t^A(i))^r}{\tilde{T}} = \frac{p^A(i)T^A}{\tilde{T}}$$

and club $i$ in league $A$ chooses $p^A(i)$ to maximize

$$\Pi^A(i) = p^A(i) \left(\psi + \Omega p^A(i) \frac{T^A}{T}\right) - c(i)K(T^A)H'(p^A(i))$$

taking $T^A, \tilde{T}$ as given. The FOC of his problem is

$$\psi + 2p^A(i)\Omega T^A \frac{T^A}{T} = c(i)K(T^A)H'(p^A(i))$$

which in a quasi-symmetric ATB equilibrium becomes (with $\tilde{T} = (t^A(i))^r + (t^B(s))^r \approx \frac{2T}{n}$ and neglecting the superscripts for the league)

$$\psi + p(i)\Omega n = c(i)K(T)H'(p(i))$$

\textsuperscript{16}Under reasonable assumptions only such equilibria exist and are stable.
Hence the quasi-symmetric equilibrium is a solution \((p(i), T)\) to

\[
\psi + p(i)n\Omega = c(i)K(T)H'(p(i)) \quad \int p(i)\mu(di) = 1 \tag{13}
\]

Analyzing (13) is not trivial in general. To proceed as simple as possible we assume \(H\) to be quadratic, i.e. \(H(p(i)) = \frac{1}{2}p(i)^2\). Then

\[
\psi + p(i)n\Omega = c(i)K(T)p(i) \quad \int p(i)\mu(di) = 1 \tag{14}
\]

**Proposition 5** Any equilibrium of (14) has the following properties \((i < j)\):

1. Relative winning chances \(p(i)/p(j)\) increase in \(\Omega\) and decrease in \(\psi\).

2. There exist uniquely determined marginal players \(v, v' \in (0, 1)\) such that

\[
\frac{\partial}{\partial \Omega} p(i) \gtrless 0 \iff v \gtrless i \tag{15}
\]

\[
\frac{\partial}{\partial \psi} p(i) \gtrless 0 \iff i \gtrless v' \tag{16}
\]

**Proof** See Appendix.

**Remark:** Whereas the "competitive balance" defined by \(\zeta(i, j) \equiv \frac{p(i)}{p(j)}\) decreases under higher \(\Omega\), the opposite holds under higher \(\psi\). This divergence originates from the fact that a change of \(\psi\) does not affect \(\zeta(i, j)\) directly (as we also know from earlier results) but has an indirect effect over \(T\). As \(\psi\) goes up, the aggregate effort to win the contests, \(T\), goes up, which (see (31)) by itself decreases \(\zeta\). Intuitively, the last effect means that relative ex-ante advantages diminish under higher aggregate effort. Taken together, these effects explain why \(\zeta\) is reduced. An increase of \(\Omega\) has the opposite effect because, other than \(\psi\), changing \(\Omega\) directly affects \(\zeta\) in a positive way.

Now, we prove that a unique solution to proposition 5 exists.

**Proposition 6** There exists a unique function \(p(i) : \mathcal{I} \to (0, \infty)\) such that (14) holds. Moreover, the aggregate effort \(T\) is uniquely determined and \(p(i)\) is at least a \(C^1\)-function.

**Proof** See Appendix.

Finally, we consider profit levels. It is straightforward to show that equilibrium profit
levels are given by
\[
\Pi(i) = \frac{1}{2} \frac{\psi^2}{c(i)K(T) - n\Omega}
\] (17)

Hence we get the following corollary (the clubs \(v, v'\) are the same as in proposition 5):

**Corollary 3** All clubs make positive profit in equilibrium. Equilibrium relative profits correspond exactly to relative chances and hence have the same comparative statics. If \(v \preceq i\) then \(\frac{\partial}{\partial \Omega} \Pi(i) \geq 0\). Further, if \(i \geq v'\) then \(\frac{\partial}{\partial \psi} \Pi(i) > 0\). Finally, aggregate profits depend only (positively) on \(\psi\) and are independent of \(\Omega\).

**Proof** See Appendix.

**Remark:** The last result means that a change of \(\Omega\) does not affect how much can be earned but it affects who earns how much. Thus, a larger Champions League prize has only distributive effect but it does not change the size of the cake. Whereas a change of \(\psi\) increases how much is earned and also affects who earns how much. Further, the sign of \(\frac{\partial}{\partial \psi} \Pi(i) > 0\) cannot be unambiguously determined\(^{17}\) if \(i < v'\).

Open question: How is average payoff affected (seems to be increasing in both; not trivial).

### 4.2 Across-League Heterogeneity and Within-League Symmetry

In this section we examine how the distribution of chances and profits between the leagues depend on the within-league prizes \(\psi_A, \psi_B\), the across-league prize \(\Omega\) and the costs. We focus entirely on the case where \(\mu\) is the continuous measure\(^{18}\) as the results do not change qualitatively otherwise (as can be expected upon the last sections).

Other than in the last section, we hold the degree of within-league heterogeneity fixed by assuming that the all clubs in a given league are symmetric (i.e., they have the same cost parameter). This means that in a symmetric equilibrium all clubs of a league choose the same effort and hence \(p^j(i) = \frac{1}{n}\) for all \(i \in \mathcal{I}\) and \(j = A, B\). However, leagues may be asymmetric, which leads to asymmetric aggregates \(T^A, T^B\) and different (overall) profit levels.

\(^{17}\)For example, if there are only two types of clubs, the low- or the high-cost type, then, depending on the parameters, the profit level might be increasing or decreasing for the low-cost type.

\(^{18}\)The advantage is that aggregates and averages are the same.
According to (11) and (12), the payoff function then becomes

\[ \Pi_j(i) = p_j(i) \left( \psi_j + \Omega \int p_{-j}(s) \frac{p_j(i) T_j}{T} \mu(ds) \right) - \frac{1}{\eta} c_j T_j p_j(i)^\eta \]

Hence the equilibrium conditions (remember \( p^j(i) = 1/n \)) are

\[ \psi_A + 2\Omega \frac{T_A^+}{T_A + T_B} = c_A T_A^\eta \]
\[ \psi_B + 2\Omega \frac{T_B^+}{T_A + T_B} = c_B T_B^\eta \] (18)

These conditions determine \((T_A, T_B)\). We assume \( c_A, c_B \geq 1, L_A, L_B > 0, \Omega \geq 0 \) and \( \eta \geq 2 \).

A solution \((T_A, T_B)\) to (18) is locally (gradient) stable if the system of differential equations,

\[ \dot{T}_A(\tau) = s_A \left( \psi_A + 2\Omega \frac{T_A^+}{T_A + T_B} - c_A T_A^\eta \right) \]
\[ \dot{T}_B(\tau) = s_B \left( \psi_B + 2\Omega \frac{T_B^+}{T_A + T_B} - c_B T_B^\eta \right) \] (19)

is locally asymptotically stable, where \( s_A, s_B > 0 \) are arbitrary adjustment speeds.

**Proposition 7** System (18) has a unique and locally stable solution \((T_A, T_B) > 0\).

**Proof** See Appendix.

**Remark:** The result on stability also implies that the mapping \( F \) (i.e., the vector field induced by the two equilibrium expressions) is a local contraction, i.e. stable under (simultaneous) tatonnement.

Using (18) we can write equilibrium profits as

\[ \Pi_A = \frac{1}{2} \left( \psi_A + \frac{c_A T_A^\eta}{\eta} (\eta - 2) \right) \]
\[ \Pi_B = \frac{1}{2} \left( \psi_B + \frac{c_B T_B^\eta}{\eta} (\eta - 2) \right) \] (20)

**Proposition 8** We have \( T_A, \Pi_A > T_B, \Pi_B \) if one of the following conditions hold:

1) \( \psi_A > \psi_B \) and \( c_A \leq c_B \)
2) \( \psi_A = \psi_B \) and \( c_A < c_B \)

**Proof** See Appendix.

**Proposition 9 (Comparative statics)** An introduction or increase of \( \Omega \) has the following effect: \( T_A, T_B, \Pi_A, \Pi_B \) all increase but
(i) if $\psi_A > \psi_B$, $c_A \geq c_B$ and $T_A > T_B$ then $\Pi_A/\Pi_B$ decreases

(ii) if $\psi_A \leq \psi_B$, $c_A < c_B$ and $T_A > T_B$ then $\Pi_A/\Pi_B$ increases

If $\Omega > 0$, increasing $\psi_j$ or decreasing $c_j$ implies that $T_j, \Pi_j$ increase, $T_{-j}, \Pi_{-j}$ decrease and $\Pi_j/\Pi_{-j}$ increases.

**Proof** See Appendix.

**Corollary 4** If $\psi_A > \psi_B$ and $c_A = c_B$ then $\Pi_A/\Pi_B$ decreases in $\Omega$. If $\psi_A = \psi_B$ and $c_A < c_B$ then $\Pi_A/\Pi_B$ increases in $\Omega$.

**Proof** See Appendix.

**Remark**: This result shows that the source of the comparative advantage (cost-driven or prize-driven) matters crucially for how a change in the Champions League prize affects the relative profits (can be seen most clearly if $\Omega = 0$ initially).

The following result shows that increasing the Champions League prize might reverse aggregate profitability compared to autarky.

**Proposition 10** Suppose that $c_A > c_B$, $\psi_A > \psi_B$ and $\psi_A/c_A > \psi_B/c_B$. Then $\Omega = 0$ implies that $T_A > T_B$ and $\Pi_A > \Pi_B$ but both $T_A/T_B$ and $\Pi_A/\Pi_B$ are decreasing functions of $\Omega$. There exists a $\Omega' > 0$ such that $T_A < T_B$ for all $\Omega > \Omega'$. Moreover, there exists $\Omega'' > \Omega'$ such that $\Pi_A < \Pi_B$ for all $\Omega > \Omega''$.

**Proof** See Appendix.

Suppose that a league has a cost advantage but a lower league prize. This proposition shows that the cost advantage crowds out the league prize disadvantage for an increasing Champions League prize. As the Champions League gets more and more important, we expect that more efficient leagues (i.e., leagues with lower costs) are more successful and differences in league prizes may play a minor role.

5 Robustness

This section provides robustness checks for our general league model developed in the preceding sections. For the robustness checks, we consider two leagues $A$ and $B$ where each league is composed out of two clubs. We denote talent investments of club $i \in \{1, 2\}$
in league \( k \in \{A, B\} \) by \( t_{ik} \). For the simulations, we consider the standard Tullock CSF, which is the most widely-used functional form of a CSF in sporting contests.\(^{19}\) The probability that club \( i \in \{1, 2\} \) in league \( k \in \{A, B\} \) wins against club \( j \in \{1, 2\} \) in league \( l \in \{A, B\} \) is given by

\[
p_{i}(t_{ik}, t_{jl}) = \frac{t_{ik}}{t_{ik} + t_{jl}}.
\]

(21)

The profit function of club \( i \) in league \( k \) is given by

\[
\pi_{ik} = \frac{t_{ik}}{t_{ik} + t_{jk}} \left[ L_{k} + \left( \frac{t_{il}}{t_{il} + t_{jl}} \frac{t_{ik}}{t_{ik} + t_{il}} + \frac{t_{jl}}{t_{il} + t_{jl}} \frac{t_{ik}}{t_{ik} + t_{jl}} \right) \Omega \right] - c_{ik}t_{ik}^{2}
\]

with \( i, j \in \{1, 2\}, i \neq j \) and \( k, l \in \{A, B\}, k \neq l \). With probability \( \frac{t_{ik}}{t_{ik} + t_{jk}} \) the club \( i \) wins against club \( j \) in league \( k \) and has the right to participate in the Champions League where this club plays with probability \( \frac{t_{il}}{t_{il} + t_{jl}} \) against the league \( l \) club \( i \) or with probability \( \frac{t_{jl}}{t_{il} + t_{jl}} \) it plays against the league \( l \) club \( j \). The respective probabilities that the league \( k \) club \( i \) wins against the league \( l \) club \( i \) and \( j \) are given by \( \frac{t_{ik}}{t_{ik} + t_{il}} \) and \( \frac{t_{ik}}{t_{ik} + t_{jl}} \), respectively.

We further assume that the league \( k \) club \( i \) incurs strictly convex talent investment costs given by the quadratic cost function \( c_{ik}t_{ik}^{2} \). We measure competitive balance as the ratio of win percentages \( CB = \frac{p_{ik}}{p_{jk}} \). The league is completely balanced (i.e., \( CB = 1 \)) if both clubs have the same win percentage, while lower or higher values of \( CB \) characterize a less balanced league with a lower degree of competitive balance.

We test robustness of the models as follows:

- In contrast to the assumption of aggregate taking behavior, we simulate the model with the classical Nash concept. We will show that our main results in the previous sections are robust as we get qualitatively similar results. Thus, the additional strategical effects under the Nash concept are second-order effects.

- In the last two subsections we separately discussed asymmetry within and across leagues. In our simulations we additionally discuss the model when the two types of asymmetry are simultaneously present.

In the following three subsection, we first simulate the model with across-league symmetry and within-league heterogeneity. Afterwards, we consider across-league hetero-

\(^{19}\)Note that the logit CSF was introduced by Tullock (1980). Skaperdas (1996) and Clark & Riis (1998) subsequently axiomatized it.
geneity but within-league symmetry. Finally, additional results are derived when both types of asymmetry are present.

5.1 Across-League Symmetry and Within-League Heterogeneity

As a first approach, we assume that the two leagues are symmetric but clubs are asymmetric within each league. Formally, the league prizes are the same across leagues, i.e., $\psi_A = \psi_B = \psi$ and the clubs have a similar cost structure across the leagues, i.e., $c_{1A} = c_{1B} = c_1$ and $c_{2A} = c_{2B} = c_2$ with $c_1 \neq c_2$. To check the robustness of Propositions 2 and 5, we simulate the effect of variations in the league prizes $\psi$ and the Champions League prize $\Omega$, respectively, on competitive balance within the two leagues. For the simulations, we use the following parameters:

<table>
<thead>
<tr>
<th>League</th>
<th>Favorite</th>
<th>Underdog</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$c_{1A} = 1$</td>
<td>$c_{2A} = 1.1$</td>
</tr>
<tr>
<td>B</td>
<td>$c_{1B} = 1$</td>
<td>$c_{2B} = 1.1$</td>
</tr>
</tbody>
</table>

Hence, in each league club 1 denotes the more efficient club which operates with lower costs as compared to club 2.

Figure 1 illustrates the effects of variations in the CL prize $\Omega$ (x-axis) on competitive balance $CB$ in both leagues (y-axis) for different values of league prizes ($\psi \in \{1, 3, 5\}$). Because the leagues are symmetric, competitive balance is the same in both leagues and therefore, it suffices to report only one league.

From Figure 1, we derive the following numerical findings regarding league prize and Champions League prize variations:

**Result 1** (i) For a positive Champions League prize, higher league prizes increase competitive balance in both leagues. If the Champions League does not exist, higher league prizes have no effect on competitive balance in either league.

(ii) A higher Champions League prize decreases competitive balance in both leagues.

(i) Consistent with Proposition 5, higher league prizes have positive effects on competitive balance in the two leagues if the Champions League exists. Moreover, consistent with Proposition 2, higher league prizes do not affect competitive balance in isolated leagues, i.e., if the Champions League does not exist ($\Omega = 0$).
(ii) Consistent with Proposition 5, a higher Champions League prize has a negative effect on competitive balance in the two leagues.

5.2 Across-League Heterogeneity and Within-League Symmetry

In this section, we assume that the clubs in the two leagues are symmetric, i.e., both clubs within a league have the same cost structure but across leagues there is asymmetry. Formally, league prizes are not the same, i.e., $\psi_A \neq \psi_B$ but the clubs have a similar cost structure within the leagues, i.e., $c_{1A} = c_{2A} = c_A$ and $c_{1B} = c_{2B} = c_B$ with $c_A \neq c_B$.

To check the robustness of Proposition 10, we simulate the effect of variations in the Champions League prize $\Omega$ on aggregate talent $T^A, T^B$ and aggregate profits $\Pi^A, \Pi^B$ in leagues $A$ and $B$, respectively. For the simulations, we use the following parameters:

<table>
<thead>
<tr>
<th>League A</th>
<th>League B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Favorite</td>
<td>$c_{1A} = 2$</td>
</tr>
<tr>
<td>Underdog</td>
<td>$c_{2A} = 2$</td>
</tr>
</tbody>
</table>

Moreover, we fix the league $B$ prize at $\psi_B = 2$ and analyze different levels of the league $A$ prize with $\psi_A \in \{3, 4, 5\}$. Hence, the assumptions of Proposition 10 are satisfied because $c_A > c_B$ and $\frac{\psi_A}{c_A} > \frac{\psi_B}{c_B}$ for all $\psi_A \in \{3, 4, 5\}$. 

Figure 1: Effects of Variations in CL Prize on CB
Figure 2 illustrates the effects of variations in the Champions League prize $\Omega$ (x-axis) on the ratio of aggregate talent $\frac{T_A}{T_B}$ (Panel a) and aggregate profits $\frac{\Pi_A}{\Pi_B}$ (Panel b) between league A and league B (y-axis) for different values of the league A prize ($\psi_A \in \{3, 4, 5\}$).

Figure 2: Effects of Variations in CL Prize on Talent and Profits

From Figure 2, we derive the following numerical findings regarding Champions League prize variations:

**Result 2**

(i) Without a Champions League prize (i.e., $\Omega = 0$), we find $T^A > T^B$ and $\Pi^A > \Pi^B$. Moreover, the ratio of aggregate talent $\frac{T_A}{T_B}$ and aggregate profits $\frac{\Pi_A}{\Pi_B}$ are decreasing functions of $\Omega$.

(ii) For each $\psi^A$, there exists $\Omega'(\psi^A)$ and $\Omega''(\psi^A)$ such that $\frac{T^A}{T^B} < 1 \forall \Omega > \Omega'(L^A)$ and $\frac{\Pi^A}{\Pi_B} < 1 \forall \Omega > \Omega''(\psi^A)$, with $\Omega''(\psi^A) > \Omega'(\psi^A)$ and $\psi_A \in \{3, 4, 5\}$.

The numerical finding is consistent with Proposition 10. Part (i) shows that if the Champions League prize is zero, aggregate talent and aggregate profits are higher in league A than in B. By increasing the Champions League prize, the ratios of $\frac{T^A}{T^B}$ and aggregate profits $\frac{\Pi_A}{\Pi_B}$ decrease regardless of the value of the league A prize. Part (ii) shows that we find critical values ($\Omega', \Omega''$) of the Champions League prize such that $T^A < T^B$ and $\Pi^A < \Pi^B$ if the Champions League prize is above these critical values. Moreover, the threshold value $\Omega''$ is always lower than $\Omega'$. For example, $\Omega''(3) = 12 > 2 = \Omega'(3)$.
5.3 Across-League Heterogeneity and Within-League Heterogeneity

In this section, we intend to verify whether one of our main results, namely the different effects on competitive balance of variations in the Champions League and domestic league prizes, also holds under full heterogeneity (i.e., across- and within-league heterogeneity).

For the simulations, we use the following cost parameters:

<table>
<thead>
<tr>
<th>League</th>
<th>Cost Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>League A</td>
<td>$c_{1A} = 1$</td>
</tr>
<tr>
<td>League B</td>
<td>$c_{1B} = 1.1$</td>
</tr>
<tr>
<td>League A</td>
<td>$c_{2A} = 1.2$</td>
</tr>
<tr>
<td>League B</td>
<td>$c_{2B} = 1.3$</td>
</tr>
</tbody>
</table>

First, we assume that the league prize between both leagues are similar, i.e., $\psi_A = \psi_B$. Here, we find qualitatively the same results as in Result 1. Second, we relax the assumption of symmetric league prizes and consider different league prizes, i.e., $\psi_A \neq \psi_B$. Specifically, we fix $\psi_B = 3$ and allow for variations in $\psi_A$.

Figure 3 illustrates the effects of variations in the league prizes $\psi$ (x-axis) on competitive balance $CB$ (y-axis) in league A (Panel a) and league B (Panel b) for different values of league A prizes ($\psi_A \in \{1, 3, 5\}$).

![Figure 3: Effects of Variations in CL Prize on CB](image)

(a) CB in League A  (b) CB in League B

From Figure 3, we derive the following numerical findings regarding Champions League and league prize variations:

**Result 3** (i) A higher Champions League prize decreases competitive balance in both leagues.
(ii) For a positive Champions League prize, a higher league A prize increase competitive balance in league A and decreases competitive balance in league B. If the Champions League does not exist, a higher league A prize has no affect on competitive balance in either league.

(i) Consistent with Proposition 5, increasing the Champions League prize has a negative effect on competitive balance in both leagues.

(ii) Consistent with Proposition 5, a unilateral increase in the league A prize produces a more balanced league A. However, in addition, we find that the competitive balance in league B decreases. Consistent with Proposition 2, without a Champions League, prize variations have no affect on competitive balance.

6 Conclusion

In this paper, we analyse a model with \( n \)-clubs in two-leagues which are connected by a superior contest, i.e., the Champions League. We show that the introduction of the Champions League has major effects on clubs’ investment decision, competitive balance and profits. The main results are as follows:

- The presence of the Champions League changes the effect of league prizes: If leagues are not connected by a Champions League, a higher league prize in both leagues has neither an effect on competitive balance nor on relative profits. However, if leagues are connected by a Champions League, a (bilateral) increase in the domestic league prizes increases competitive balance within the leagues.

- Simulations show that a unilateral increase in the league prize in league A increases competitive balance in league A while it decreases competitive balance in league B due to spill-over effects.

- Our model shows that the Champions League and league prizes have different effects on competitive balance: an increase in the Champions League prize decreases competitive balance.

- Aggregate profits are increasing in the league prizes but are independent of the Champions league prize. The size of the Champions League prize has only distributive effects within leagues but it does not change the size of the cake.
• Further results are derived with respect to the existence and uniqueness of the solution.

This paper can be seen as a first approach to analyze the interaction effects of leagues which are integrated in an overall contest.
A Appendix

A.1 Proof of Proposition 1

Because of zero homogeneity: \( p(i) = P\left(\frac{e(i)}{T}\right) \equiv P\left(\frac{e(i)}{T}\right) \) in slight abuse of notation. By assumption, we have \( P' > 0 \) and \( P'' \leq 0 \). Hence \( e(i) = \varphi(p(i)) T \), where \( \varphi(x) \equiv P^{-1}(x) \), and thus \( \varphi' > 0 \) and \( \varphi'' \geq 0 \). The rest is obvious.

A.2 Proof of Theorem 1

\( L \) is homogeneous, since \( H \) is homogeneous. Therefore, (4) implies \( p(i) = L\left(\frac{c(j)}{c(i)}\right)p(j) \) for all \( i, j \), where \( L \equiv H'^{-1} \). Hence by (4)

\[
p(i) = \left( \int L\left(\frac{c(i)}{c(s)}\right)\mu(ds) \right)^{-1} \tag{22}
\]

for all \( i \in \mathcal{I} \). From (22) we get

\[
p'(i) = -p(i)^2 \int L'\left(\frac{c(i)}{c(s)}\right)\frac{c'(i)}{c(s)}\mu(ds) \leq 0 \tag{23}
\]

The rest then follows immediately.

A.3 Proof of Corollary 1

The first claim follows immediately from (23) and the second is an envelope theorem result:

\[
\Pi'(i) = -K(T)c'(i)H\left(\frac{p(i)}{H'(p(i))}\right) < 0
\]

A.4 Proof of Proposition 2

Using (4) we get

\[
\Pi(i) = R\left( p(i) - \frac{H(p(i))}{H'(p(i))} \right) \tag{24}
\]
Hence $\Pi(i) > 0 \ \forall i$ by the strict convexity of $H$ and $H(0) = 0$. According to (22), $p(i)$ and $p(i)/p(j)$ are independent of $R$.

Using (4) and the fact that $p_R(i; R) = 0$ gives

$$T'(R) = \frac{K(T)}{RK'(T)} > 0 \quad (25)$$

Further, the envelope theorem gives

$$\Pi_R(i; R) = p(i) - c(i)K'(T)H(p(i))T'(R)$$

Using (25) shows that $\Pi_R(i; R) > 0$ iff $\Pi(i) > 0$. Finally, (24) shows that $\Pi(i)/\Pi(j)$ is independent of $R$.

A.5 Proof of Proposition 3

Imitating the steps in the proof of theorem 1, we immediately see that (22) and (23) hold again. Thus $p(i)$ is independent of $\psi$. Further

$$\int \frac{R(\psi, T)}{K(T)} \mu(ds) = \int c(s)H'(p(s)) \mu(ds) \quad (26)$$

where the RHS is a number. Hence the Inada condition (5) and continuity of $\frac{R(\psi, T)}{K(T)}$ together imply existence of an aggregate $T$ that solves (26), and (6) implies uniqueness. The last claim follows as the LHS is a strictly monotonic function.

A.6 Proof of Corollary 2

The first claims follow from (23) and the Envelope-theorem. The fact that $\frac{\partial p(i)}{\partial \psi} = 0$ follows from (22). Then (26) implies

$$T'(\psi) = \frac{R_\psi K(T)}{RK'(T) - RTK(T)} \quad (27)$$
and (6) implies $T'(\psi) > 0$. Equilibrium profits are

$$\Pi(i) = c(i)K(T)(p(i)H'(p(i)) - H(p(i)))$$

Hence $\frac{\partial}{\partial\psi}\Pi(i) > 0$ and $\Pi(i)/\Pi(j)$ is independent of $\psi$.

A.7 Proof of Proposition 4

(9) has the form $\phi(p(i)) = M(i,p(i))$. By SOC we have $\phi'(p(i)) - M_1(i,p(i)) < 0$, hence

$$p'(i) = \frac{c'(i)K(T)H'(p(i))}{\phi'(p(i)) - M_1(i,p(i))} < 0$$

as $c'(i) > 0$, and $\Pi'(i) < 0$ again follows from the Envelope theorem.

A.8 Proof of Proposition 5

From (14) we get

$$p(i) = \frac{\psi}{c(i)K(T) - n\Omega}$$

(28)

and

$$1 = \int \frac{\psi}{c(s)K(T) - n\Omega} \mu(ds)$$

(29)

Using the IFT on (29) gives

$$T'(\Omega) = \frac{n}{K'(T)} \frac{\int \frac{1}{\xi(s)} \mu(ds)}{\int \frac{c(s)}{\xi^2(s)} \mu(ds)}$$

(30)

where $\xi(i) \equiv c(i)K(T) - n\Omega$, and $\xi(i) > 0$ as $p(i) > 0$. From (28) we get $\frac{\partial}{\partial\Omega}p(i) > 0$ iff $n - c(i)K'(T)T'(\Omega) > 0$. Using (30), this inequality becomes

$$f(i) \equiv \int \frac{\frac{c(s)}{\xi(s)} - \frac{c(i)}{\xi^2(s)}}{\xi(s)} \mu(ds) > 0$$

But as

$$f'(i) = \int \frac{-\frac{c'(i)}{\xi(s)} \mu(ds)}{\xi(s)} < 0$$
and $f(0) > 0$ and $f(1) < 0$ there must exist a unique $v \in (0, 1)$ such that $f(v) = 0$, which proves (15).

A similar proceeding gives

$$T'(L) = \frac{1}{\psi'K'(T)} \int \frac{1}{\xi(s)} \mu(ds) \int \frac{c(s)}{\xi(s)^2} \mu(ds)$$

From (28) and using the last result we get $\frac{\partial}{\partial \psi} p(i) > 0$ iff

$$c(i)K(T) - n\Omega > \int \frac{c(i)}{\xi(s)} \mu(ds) \int \frac{c(s)}{\xi(s)^2} \mu(ds)$$

which is equivalent to

$$g(i) = \int \frac{c(s)\xi(i) - c(i)\xi(s)}{\xi(s)^2} \mu(ds) > 0$$

As

$$g'(i) = \int \frac{c'(i)n\Omega}{\xi(s)^2} \mu(ds) > 0$$

and

$$g(0) = \int \frac{n\Omega (c(0) - c(s))}{\xi(s)^2} \mu(ds) < 0$$

$$g(1) = \int \frac{n\Omega (c(1) - c(s))}{\xi(s)^2} \mu(ds) > 0$$

there exists a unique $v'$ such that $g(v') = 0$, which proves (16).

Define

$$\zeta(i, j) \equiv \frac{p(i)}{p(j)} = \frac{c(j)K(T) - n\Omega}{c(i)K(T) - n\Omega}$$

(31)

Taking the derivative of $\zeta$ with respect to $\Omega$ and rearranging gives $\frac{\partial}{\partial \Omega} \zeta(i, j) > 0$ iff $K(T) - \Omega K''(T)T'(\Omega) > 0$. Plugging in (30) gives

$$K(T) - \Omega n \int \frac{1}{\xi(s)} \mu(ds) \int \frac{c(s)}{\xi(s)^2} \mu(ds) > 0$$
which holds as \( p(i) > 0, c(i) \geq 1 \) and
\[
\frac{1}{\xi(i)} \int c(s) \mu(ds) \over \frac{1}{\xi(i)^2} \mu(ds) < 1
\]

Proceeding along the same line yields \( \frac{\partial}{\partial \psi} \zeta(i, j) < 0 \), which completes the proof.

\[\blacksquare\]

A.9 Proof of Proposition 6

Note that the cost level \( K \) is a solution to \( G(K) \equiv \int_{0}^{1} \frac{1}{c(s)K-n\Omega} \mu(ds) = 1/\psi \). As \( p(i) > 0 \) by presumption, we must have \( K > n\Omega \). As \( c(i) \in [1, \bar{c}] \) there must be values \( K_0, K_1 \) such that \( G(K_0) > 1/\psi > G(K_1) \). As \( c' > 0 \) we also have \( G'(K) < 0 \). By the continuity of \( G \) there exists thus a unique \( K \) that solves the above equation. As \( K' > 0 \) the aggregate \( T \) also is uniquely determined. But then (28) immediately shows that there exists exactly one function \( p(i) \) which inherits the claimed property from \( c(i) \).

A.10 Proof of Corollary 3

The last claim follows from \( \int \Pi(i) \mu(di) = \frac{\psi}{2} \) and the first two claims are obvious from (17). The claim on \( \frac{\partial}{\partial \Omega} \Pi(i) \) follows from proposition 5 and the fact that \( \Pi(i) = \frac{1}{2} \psi p(i) \).

Finally, it is easy to show that \( \frac{\partial}{\partial \psi} \Pi(i) > 0 \) holds iff
\[
\frac{1}{\xi(i)} \int \frac{c(s)}{\xi(s)^2} \mu(ds) + n\Omega \int \frac{c(i) - c(s)}{\xi(s)^2} \mu(ds) > 0
\]
holds, which certainly is the case if \( i \geq v' \).

\[\blacksquare\]

A.11 Proof of Proposition 7

Suppose \((T_A, T_B) > 0\) is an equilibrium. System (19) is stable if \( Det(H) \) the determinant of the derivative matrix of the RHS of (19), is positive and its trace negative, i.e. \( Trace(H) < 0 \). We get
\[
Det(H) = \frac{s_A s_B \eta \left(c_A T_A^n c_B T_B^n (T_A + T_B)^2 \eta - 2 \Omega T_A T_B (c_A T_A^n + c_B T_B^n)\right)}{T_A T_B (T_A + T_B)^2} \tag{32}
\]
Hence $\text{Det}(H) > 0$ iff

$$c_A T_A^n c_B T_B^n (T_A + T_B)^2 \eta > 2 \Omega T_A T_B (c_A T_A^n + c_B T_B^n)$$

Using (18) this condition can be rephrased as

$$c_A T_A^{n+1} c_B T_B^n \eta + c_B T_B^{n+1} c_A T_A^n \eta > c_A T_A^n (c_B T_B^n - L_B) + c_B T_B^{n+1} (c_A T_A^n - L_A)$$

which is satisfied by the assumptions made. To see that $\text{Tr}(H) < 0$ note that the first diagonal entry of $H$, $H_{11}$, is

$$H_{11} = s_A \left( \frac{2 \Omega T_B}{(T_A + T_B)^2} - c_A \eta T_A^{n-1} \right) = \frac{s_A}{T_A} \left( c_A T_A^n \left( \frac{T_B}{T_A + T_B} - \eta \right) - \psi_A \frac{T_B}{T_A + T_B} \right) < 0$$

where the second equality uses (18). Similarly, we get $H_{22} < 0$ and hence $\text{Tr}(H) < 0$.

For existence and uniqueness note that the two equilibrium expressions

$$F_A (T_A, T_B) = \psi_A + 2 \Omega \frac{T_A}{T_A + T_B} - c_A T_A^n$$
$$F_B (T_A, T_B) = \psi_B + 2 \Omega \frac{T_B}{T_A + T_B} - c_B T_B^n$$

induce a vector field

$$F = \begin{pmatrix} F_A \\ F_B \end{pmatrix} : \begin{pmatrix} T_A \\ T_B \end{pmatrix} \to [0, \infty)^2$$

From the assumptions it is easy to see that there always exists $\varepsilon > 0$ such that the vector field points into the interior at boundary points $T_j = \varepsilon$, $j = A, B$. As the determinant of $F$ at critical points must be positive (this follows from the stability analysis above) existence and uniqueness of the equilibrium follows from the Index theorem (see e.g. Vives (2001) or Hefti (2011)).

A.12 Proof of Proposition 8

Rather than attempting a direct proof, the result on $T_A, T_B$ can be obtained from proposition 5 of Hefti (2011). Suppose the game is symmetric, i.e. $\psi_A = \psi_B = \psi$ and $c_A = c_B = c$. It is then easy to verify that $T'(\psi) > 0$ and $T'(c)$, hence the game is monotonic in these
parameters. Thus it suffices to verify that the symmetric game has no asymmetric equilibrium, which holds if \( T'_A(T_B) > -1 \). Hence we must show

\[
T_A'(T_B) = \frac{2\Omega \frac{T_A}{(T_A+T_B)^2}}{2\Omega \frac{T_B}{(T_A+T_B)^2} - \eta c_A T_A^\eta - 1} > -1
\]

Using (18) it is easy to show that the denominator is negative and hence aggregates are strategic substitutes. We therefore need to verify that

\[
2\Omega \frac{T_A}{(T_A+T_B)^2} < \eta c_A T_A^\eta - 1 - 2\Omega \frac{T_B}{(T_A+T_B)^2}
\]

which is equivalent to (use (18))

\[
2\Omega \frac{T_A}{T_A + T_B} < \eta c_A T_A^\eta
\]

But this condition holds as (use (18) again) it is equivalent to

\[-\psi_A < c_A T_A^\eta (\eta - 1)\]

To see the result for profits note that e.g.

\[
\Pi_A = \frac{\eta - 1}{\eta} \left( \psi_A + \Omega \frac{T_A}{T_A + T_B} \right)
\]

Hence if 1) holds, we have \( \psi_A > \psi_B \) and also \( T_A > T_B \), thus \( \Pi_A > \Pi_B \). 2) is shown similarly.

\[\blacksquare\]

A.13 Proof of Proposition 9

First, note that (18) implies that

\[
T_A \psi_B < T_B \psi_A \iff c_B T_B^\eta - 1 < c_A T_A^\eta - 1
\]

as well as

\[
\frac{\psi_A - \psi_B}{2\Omega} > \frac{T_B - T_A}{T_A + T_B} \iff c_B T_B^\eta < c_A T_A^\eta
\]
Next, using the IFT, the comparative statics with respect to (33) are

\[
\left( \frac{dT_A}{dT_B} \right) = \frac{1}{Det(H)} \begin{pmatrix}
  c_B T_B^{-1} \eta - \frac{2\Omega T_A}{(T_A + T_B)^2} & -\frac{2\Omega T_A}{(T_A + T_B)^2}
  \\
  -\frac{2\Omega T_B}{(T_A + T_B)^2} & c_A T_A^{-1} \eta - \frac{2\Omega T_B}{(T_A + T_B)^2}
\end{pmatrix}
\begin{pmatrix}
  \frac{\partial F_A}{\partial x} dx \\
  \frac{\partial F_B}{\partial x} dx
\end{pmatrix}
\]

(36)

where \( Det(H) \) is the Determinant of the derivative matrix of (33). As this matrix is similar to the derivative matrix of (19) (for \( s_A = s_B = 1 \)) we know from the proof of proposition 7 that \( Det(H) > 0 \) and hence (36) is well-defined.

The effects of an increase of \( \psi_A \) are clear from (36) and (20). Also \( c_A \uparrow \) implies \( T_A \downarrow \) and \( T_B \uparrow, \Pi_B \uparrow \).

\[
T_A'(\Omega) = \frac{2T_A}{Det(H)(T_A + T_B)^2T_B} (c_B T_B^\eta \eta(T_A + T_B) - 2\Omega T_B)
\]

(37)

Plugging in (32) gives

\[
T_A'(\Omega) = \frac{2T_A^2 (c_B T_B^\eta \eta(T_A + T_B) - 2\Omega T_B)}{\eta (c_A T_A^\eta c_B T_B^\eta(T_A + T_B)^2 \eta - 2\Omega T_A T_B (c_A T_A^\eta + c_B T_B^\eta))}
\]

where the denominator is positive. Using (18) it is easy to verify that the nominator is also positive. Hence \( T_A'(\Omega), T_B'(\Omega) > 0 \). From (20) we immediately get \( \Pi_A'(\Omega), \Pi_B'(\Omega) > 0 \).

Relative profits are

\[
\frac{\Pi_A}{\Pi_B} = \frac{\psi_A + \frac{\eta - 2}{\eta} c_A T_A^\eta}{\psi_B + \frac{\eta - 2}{\eta} c_B T_B^\eta}
\]

Hence relative profits increase iff

\[
(\eta - 2) c_A T_A^{\eta - 1} \left( L_B + \frac{\eta - 2}{\eta} c_B T_B^\eta \right) T_A'(x) > (\eta - 2) c_B T_B^{\eta - 1} \left( \psi_A + \frac{\eta - 2}{\eta} c_A T_A^\eta \right) T_B'(x)
\]

For \( x = \Omega \) the claim in the proposition holds if

\[
\frac{c_B T_B^\eta \eta(T_A + T_B) - 2\Omega T_B}{c_A T_A^\eta \eta(T_A + T_B) - 2\Omega T_A} < \frac{c_B T_B^{\eta + 1} (\psi_A + \frac{\eta - 2}{\eta} c_A T_A^\eta)}{c_A T_A^{\eta + 1} (\psi_B + \frac{\eta - 2}{\eta} c_B T_B^\eta)}
\]

(38)

If \( c_B T_B^{\eta - 1} \leq c_A T_A^{\eta - 1} \) (which holds under (i)) then a sufficient condition for (38) to hold is that

\[
\frac{c_B T_B^\eta \eta(T_A + T_B)}{c_A T_A^\eta \eta(T_A + T_B)} < \frac{c_B T_B^{\eta + 1} (\psi_A + \frac{\eta - 2}{\eta} c_A T_A^\eta)}{c_A T_A^{\eta + 1} (\psi_B + \frac{\eta - 2}{\eta} c_B T_B^\eta)}
\]

33
which reduces to (use 9 again) $T_A L_B < T_B L_A$, which holds by (34) and (i). To see (ii), proceed analogously. The claim holds if (38) holds with reversed sign. A similar line of arguments establishes that this new inequality is true if $c_B T_B^{\eta-1} > c_A T_A^{\eta-1}$, which holds under (34) (reverse signs) and (ii).

A.14 Proof of Corollary 4

Because $\psi_A > \psi_B$ and $c_A = c_B$ implies $T_A > T_B$ the result follows from proposition 9. The second claim follows similarly.

A.15 Proof of Proposition 10

For $\Omega = 0$ we get $T_A > T_B$ from (18) and $\Pi_A > \Pi_B$ then follows immediately. $T_A/T_B$ increases if $T_A'/T_B' > T_A/T_B$. Using (37) this holds if $c_B T_B^{\eta-1} > c_A T_A^{\eta-1}$. Hence if $T_A \geq T_B$, then $c_A > c_B$ directly implies that $T_A/T_B$ decreases in $\Omega$. If $T_A < T_B$ then (34) implies that $T_A/T_B$ decreases in $\Omega$ and hence $T_A/T_B$ is a decreasing function of $\Omega$. Let $\Omega' > 0$ be the solution of $\psi_A + \Omega c_A = \psi_B + \Omega c_B$. Then, by uniqueness of the equilibrium, it is easy to see from (18) that $T_B(\Omega') = T_B(\Omega')$. As $T_A/T_B$ is decreasing in $\Omega$ this implies that $T_A/T_B < 1$ for all $\Omega > \Omega'$. From the proof of proposition 9 we know that $\Pi_A/\Pi_B$ is a decreasing function of $\Omega$ if $T_A \psi_B < T_B \psi_A$, which holds if $T_A \geq T_B$. Similarly, we get the same result if $T_A < T_B$, as this implies that $T_A \psi_B < T_B \psi_A$ (use (34)).

Further, if $\Omega \geq \psi_A + 2\Omega - T_A/T_B$, we get $\Pi_B > \Pi_A$. To see this, note that $\psi_A + \psi_B + 2\Omega = c_A T_A^{\eta} + c_B T_B^{\eta}$. Using this in (20) shows that $\Pi_B > \Pi_A$ iff $\psi_B + \frac{\eta-2}{\eta} c_B T_B^{\eta} > \psi_A + \frac{\eta-2}{\eta} c_A T_A^{\eta}$ which, using the last equality and $\Omega > 0$, is equivalent to

$$\psi_B \left(1 + \frac{\eta}{\eta - 2}\right) + 2\Omega - \frac{\eta}{\eta - 2} > \psi_A \left(1 - \frac{\eta}{\eta - 2}\right) + \frac{2\eta}{\eta - 2} c_A T_A^{\eta}.$$  

Hence, $\Omega \geq c_A T_A^{\eta}$ is sufficient for $\Pi_B > \Pi_A$. This condition itself holds iff $\Omega \geq \psi_A + 2\Omega - T_A/T_B$, as claimed.

Now, let $(\Omega_n)$ be an increasing sequence. To prove the claim in the proposition we
have to show that $\Omega_n \left(1 - 2 \frac{T_A}{T_A + T_B}\right) \geq \psi_A$ if $n$ is sufficiently large. Write

$$\Omega_n \left(1 - 2 \frac{T_A (\Omega_n)}{T_A (\Omega_n) + T_B (\Omega_n)}\right) = \Omega_n (1 - 2r_n)$$

If $\Omega_n > \Omega'$ then $r_n < 1/2$. As $r_n$ is a decreasing and bounded sequence it is also convergent and $\lim r_n \in [0, 1/2]$. Therefore $\Omega_n (1 - 2r_n)$ is an increasing sequence and $\lim_{n \to \infty} \Omega_n (1 - 2r_n) = \infty$. Thus $\exists \Omega''$ such that $\Pi_A < \Pi_B \forall \Omega > \Omega''$.

As for $\Omega = \Omega'$ we have $T_A = T_B$, expression (20) implies that $\Pi_A (\Omega') > \Pi_B (\Omega')$, which gives $\Omega'' > \Omega'$.

Note that $\Omega_0$ is determined by $\frac{\psi_A + \Omega_0}{c_A} = \frac{\psi_B + \Omega_0}{c_B}$.

$\blacksquare$
References


